

LOW DIMENSIONAL PROJECTIVE INDECOMPOSABLE MODULES FOR CHEVALLEY GROUPS IN DEFINING CHARACTERISTIC

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ABSTRACT. The paper studies lower bounds for the dimensions of projective indecomposable modules for Chevalley groups G in defining characteristic p . The main result extending earlier one by Malle and Weigel (2008) determines the modules in question of dimension equal to the order of a Sylow p -subgroup of G . We also substantially generalize a result by Ballard (1978) on lower bounds for the dimensions of projective indecomposable modules and find lower bounds in some cases where Ballard's bounds are vacuous.

1. INTRODUCTION

Let G be a finite group and p a prime number. Let F be a field of characteristic p dividing the order $|G|$ of G . We always assume that F contains a primitive m -root of unity, where $m = |G|/|G|_p$ and $|G|_p$ is the order of a Sylow p -subgroup U of G . The group algebra FG of G over F viewed as a left FG -module is called the regular module and indecomposable direct summands are called *principal indecomposable FG -modules*, customarily abbreviated as PIM's. These are classical objects of study in the modular representation theory of finite groups [3]. One of the main open problems is to determine their dimensions or at least provide satisfactory information on the dimensions.

This paper studies this problem for finite Chevalley groups in defining characteristic p . For certain groups of small rank the PIM dimensions have been explicitly computed, but not much is known in general. A rather detailed survey of the current state of the problem and the methods used to attack it is done by J. Humphreys [27].

The results of this paper concentrate mainly on obtaining lower bounds for the PIM dimensions. The absolute lower bound for a PIM dimension for any finite group G is $|G|_p$, and this bound is attained for every Chevalley group. Indeed, every such group has an irreducible FG -module of dimension $|G|_p$, hence of p -defect 0, known as the Steinberg module. This is unique if G is quasi-simple, and this is a PIM. Our use of the term 'lower bound' assumes that we exclude irreducible modules of p -defect 0.

The earliest result on lower bounds for PIM dimensions for Chevalley groups is due to Ballard [1, Corollary 5.4], who considered only non-twisted groups. His result is stated in the same shape in [27, §9.7]. We show in Section 4 how to extend Ballard's result for arbitrary twisted groups, and we also obtain a version for it for a parabolic subgroup in place of a Borel subgroup in the original Ballard's statement.

Ballard's lower bound is not available for every PIM, for instance, it is useless for any PIM for any non-twisted Chevalley group over the field of order 2. Therefore, it is essential

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to make it clear when Ballard's type bound is applicable, and this is desirable to be made in terms of the standard parameterization of PIM's, specifically, in terms of their socles. Recall that every PIM (for any finite group G) has an irreducible socle, which determines the PIM. This establishes a bijection between PIM's and the irreducible representations of G , which we refer to here as the standard parametrization. The PIM corresponding in this way to the trivial FG -module 1_G is called 1-PIM in [34].

No further result was known over almost 30 years until Malle and Weigel [34] determined the 1-PIM's of dimension $|G|_p$. They did so for all simple groups G and for all primes dividing $|G|$. For Chevalley groups in characteristic p they suggested a method called the parabolic descent in [34]. This allows to bound from below the 1-PIM dimension in terms of Levi subgroups of parabolic subgroups of G . The method in its original shape does not work for other PIM's. In this paper we develop the method further to a level when it can be used for arbitrary PIM's (in characteristic p). This allows to extend the above mentioned result by Malle and Weigel [34] as follows:

Theorem 1.1. *Let G be a quasi-simple Chevalley group in defining characteristic p , and let Φ be a p -modular PIM of non-zero defect. Then $\dim \Phi > |G|_p$, unless Φ is a 1-PIM and $G/Z(G) \cong PSL(2, p)$ or ${}^2G_2(3) \cong \text{Aut } SL(2, 8)$.*

The parabolic descent reduces the proof to groups of BN-pair rank at most 2, and for most of them Theorem 1.1 is already known to be true. However, for the groups $G = SU(4, p)$, ${}^3D_4(p)$ and ${}^2G_2(3^k)$ the PIM dimensions are not known, although the character tables are known. These are not sufficient to rule out the above three groups, and the parabolic descent method is only partially helpful. In order to prove Theorem 1.1 for these groups we also make use of the following simple observation (Lemma 3.12): $\dim \Phi = |G|_p \cdot (\chi, 1_U^G) = |G|_p \cdot (\chi, \Gamma)$, where χ is the character of Φ , Γ is a Gelfand-Graev character and U is the Sylow p -subgroup of G . (Here 1_U is the trivial representation of U and 1_U^G is the induced representation.)

I conjecture that for classical groups G of rank n the dimension of a PIM other than the Steinberg one is at least $(n-1) \cdot |G|_p$. Some progress is achieved in this paper by using a new idea based on the analysis of common irreducible constituents of the ordinary character of a PIM and the induced module 1_U^G , where U is a Sylow p -subgroup of G . Let W be the Weyl group of G viewed as a group with BN-pair. In favorable circumstances, in particular, for groups $SL(n, q)$, $n > 4$, $E_6(q)$, $E_7(q)$, $E_8(q)$ the PIM dimension is shown to be at least $d \cdot |G|_p$, where d is the minimum dimension of a non-linear irreducible representation of W (Theorem 6.4). If $G = SL(n+1, q)$, $n > 3$ then the rank of G is n , $W \cong S_{n+1}$, the symmetric group, and $d = n$. So in this case the conjecture is confirmed. Note that if $q = 2$ then there is a PIM of dimension $n \cdot |G|_p$; if $q > 2$ and $n > 2$ then there is a PIM of dimension $(n+1) \cdot |G|_p$ [43].

Notation. \mathbb{Q} , \mathbb{C} are the rational and complex number fields, respectively, and \mathbb{Z} is the ring of integers. F_q is the finite field of q elements, and F an algebraically closed field of characteristic $p > 0$.

If G is a finite group, then $Z(G)$ is the center and $|G|$ is the order of G . If p is a prime then $|G|_p$ is the p -part of $|G|$ and also the order of every Sylow p -subgroup of G . A

p' -group means a finite group with no element of order p . If $g \in G$ then $|g|$ is the order of g . Notation for classical groups are standard.

All representations and modules are over F or over \mathbb{C} (unless otherwise stated). Sometimes we deal with RG -modules, where $1 \in R \subset \mathbb{C}$ is a subring. In this case all RG -modules are assumed to be free as R -modules. If the ground field is clear from the context, we take liberty to use the term ' G -module'. All modules are assumed to be finitely generated. Representations of G in characteristic p are called p -*modular*, and those over the complex numbers are called *ordinary*. The regular representation of G is denoted by ρ_G^{reg} , and the trivial one-dimensional representation is denoted by 1_G . We also use 1_G to denote the trivial one-dimensional module and its (Brauer) character. If M is an FG -module, then $\text{Soc } M$ means the socle of M , the sum of all irreducible submodules.

The set of irreducible characters of G is denoted by $\text{Irr } G$, and $\mathbb{Z}\text{Irr } G$ is the \mathbb{Z} -span of $\text{Irr } G$; elements of $\mathbb{Z}\text{Irr } G$ are called *generalized characters*.

A projective indecomposable FG -module is called a PIM and usually denoted by Φ . Every PIM is determined by its socle. The PIM whose socle is 1_G is called here 1-PIM, and denoted by Φ_1 . More notation concerning projective modules is introduced in Section 3. We set $c_\Phi = \dim \Phi / |G|_p$.

If χ is a character vanishing at all non-identity p -elements then we write $c_\chi = \chi(1)/|G_p|$; this is an integer.

Let M be an FG - or RG -module. We set $C_M(G) = \{m \in M : gm = m \text{ for all } g \in G\}$. Thus, $C_M(N)$ is the set of G -invariants (or fixed points) on M .

Let H be a subgroup of G and N a normal subgroup of H . Then $C_M(N)$ is an H -module, and when it is viewed as an H/N -module, it is denoted by $\overline{M}_{H/N}$ (or \overline{M}), and called the *Harish-Chandra restriction of M to H/N* . Conversely, given an H/N -module D , one can view it as an H -module with trivial action of N . Then the induced G -module D^G (when D is viewed as an H -module) is denoted by $D^{\#G}$ and called *Harish-Chandra induced from D* . For details see [13, p. 667-668, §70A], where these operations are called generalized restriction and induction. This corresponds to similar operations on characters and Brauer characters. The Harish-Chandra restriction and induction extends by linearity to the class functions on G . So if χ is a class function on G then $\overline{\chi}_{H/N}$ is the corresponding Harish-Chandra restriction of χ to H/N , and if λ is a class function on H/N then $\lambda^{\#G}$ denotes the Harish-Chandra induced class function on G . (For the ordinary induction we use notation λ^G .) Let $\eta : G \rightarrow \mathbb{C}$ be a class function on G . The formula $(\lambda^{\#G}, \eta) = (\lambda, \overline{\eta}_{H/N})$ is an easy consequence of the Frobenius reciprocity and called the *Harish-Chandra reciprocity*. (This is formula 70.1(iii) in [13, p.668].) Note that $\overline{\eta}_{H/N}$ can be obtained as the truncation of η . This is defined by $\frac{1}{|N|} \sum_{n \in N} \eta(hn)$ for $h \in H$ viewed as a function on H/N .

Let \mathbf{G} be a connected reductive algebraic group. An algebraic group homomorphism $\mathbf{G} \rightarrow \mathbf{G}$ is called *Frobenius* if its fixed point subgroup is finite. We denote Frobenius endomorphisms by Fr . So $\mathbf{G}^{Fr} = \{g \in \mathbf{G} : Fr(g) = g\}$ is a finite group, called here a finite reductive group. If \mathbf{G} is simple and simply connected, we refer to $G = \mathbf{G}^{Fr}$ as a *Chevalley group*. More notation concerning algebraic groups will be introduced in Section 5. If G is a finite reductive group or an algebraic group, p is reserved for the defining characteristic of G .

2. THE METHODS AND MAIN RESULTS

In order to guide a reader through the paper, we comment here the machinery used in the proofs.

2.1. Parabolic descent. A well known standard fact on PIM's for any finite group G is that the mapping $\Phi \rightarrow \text{Soc } \Phi$ yields a bijection $\text{PIM}_G \rightarrow \text{Irr } G$ between the set of PIM's and the set of equivalence classes of irreducible FG -modules. In addition, every projective module is a sum of PIM's. Therefore, every projective module is determined by its socle, and a projective module is a PIM if and only if its socle is irreducible.

Let G be a Chevalley group, so $G = \mathbf{G}^{Fr}$, where \mathbf{G} is a simply connected simple algebraic group. The following result is the well known Smith-Dipper theorem (see [36], [17], [4], [21, 2.8.11]):

Lemma 2.1. *Let G be a group with a split BN -pair in characteristic p , P a parabolic subgroup with Levi subgroup L . Let V be an irreducible FG -module. Then $\overline{V}_L = C_V(O_p(P))|_L$ is an irreducible L -module.*

In other words, for every parabolic subgroup P of G there is a mapping $\sigma_{G,P} : \text{Irr } G \rightarrow \text{Irr } L$ defined via the Harish-Chandra restriction $V \rightarrow \overline{V}_L$. Note that $C_V(O_p(P))$ coincides with $\text{Soc } V|_P$ as $p = \text{char } F$. The mapping $\sigma_{G,P}$ is surjective (see Lemma 5.1). This allows one to define a surjective mapping $\pi_{G,P} : \text{PIM}_G \rightarrow \text{PIM}_L$ as the composition of the mappings

$$\Phi \rightarrow \text{Soc } \Phi \rightarrow \overline{(\text{Soc } \Phi)}_L \rightarrow \Psi,$$

where Ψ is the PIM for L with socle $\overline{(\text{Soc } \Phi)}_L$. This is well defined in view of Lemma 2.1. We call this mapping the *parabolic descent from G to L* . (We borrow the term from [34] but the meaning of the term is not the same as in [34].) One observes that $\pi_{G,L}(\Phi)$ is a direct summand of $\overline{\Phi}_L = C_\Phi(O_p(P))|_L$ (but the equality rarely holds). This implies that $c_\Phi \geq c_\Psi$, where $\Psi = \pi_{G,P}(\Phi)$, see Lemma 3.10. In the special case where $\Phi = \Phi_1$ is the 1-PIM, this fact has been exploited in [34]. An attempt to extend it to other PIMs meets an obstacle, specifically, in order the method could work one needs at least to guarantee that Ψ is not a defect zero irreducible FL -module. We show how to manage with this difficulty in Section 5.

A weak point of the parabolic descent is that it only allows to bound (from below) $\dim \Phi$ in terms of $\dim \Psi$, and can not be used for showing that c_Φ grows as the rank of G tends to infinity. Nonetheless this is useful for classifying PIM's of dimension $|G|_p$, which is one of our tasks below.

A formal analog of Lemma 2.1 for a projective module would be a claim that if Φ is a PIM for G then $\overline{\Phi}_L$ is a PIM for L . However, this is not true. This is evident from Propositions 4.5 and 4.7.

2.2. Ballard's bound revised. Let B, N be the subgroups of G defining the BN -pair structure on G (see [13, §69.1]). The group $T := N \cap B$ is normal in N . Set $W = N/T$. Then B is a Borel subgroup of G , $U = O_p(B)$ is a Sylow p -subgroup of G and $T = N \cap B$

is a maximal torus of B . Let Φ be a PIM for G with socle M . Ballard [1, Corollary 5.4] proves that $\dim \Phi \geq |W\beta| \cdot |G|_p$, equivalently, $c_\Phi \geq |W\beta|$ in our notation, where $|W\beta|$ is the size of the W -orbit of the (Brauer) character β of T afforded by the restriction of $\text{Soc}(M|_B)$ to T . He assumes G to be non-twisted. In Section 4 Ballard's result is generalized to all twisted groups as follows. The conjugacy action of N on T induces an action on $\text{Irr } T$, and for $\beta \in \text{Irr } T$ let $|N\beta|$ denote the size of the N -orbit of β . Let Φ be a PIM for G with socle V and let β be the Brauer character of the FG -module $C_V(U)|_T$; it is well known that $\beta \in \text{Irr } T$. Then our version of Ballard's result states that $c_\Phi \geq |N\beta|$ (Proposition 4.5).

The bound $|N\beta|$ is vacuous if $|N\beta| = 1$. For instance, there are PIM's for which $\beta = 1_T$, so the bound is vacuous for such PIM's. In addition, in many cases $|N\beta|$ is small, and the bound is not sharp. (This happens for instance if $|N/T| = 2$ but G is not $SL(2, q)$.)

We find out that the nature of Ballard's result is not specific for PIM's. We prove a similar result for arbitrary characters vanishing at all non-trivial p -elements, see Proposition 4.4. This implies the result for PIM's as their characters have this property.

The parabolic descent can be combined with our interpretation of the Ballard bound as follows. Let P be a standard parabolic subgroup of G [13, 65.15], and let L be the standard Levi subgroup of P [13, 69.14]. Let Ψ be the parabolic descent of Φ . Let N_L be the normalizer of L in N , so N_L acts on L via conjugation. For $n \in N_L$ let Ψ^n denote the twist of Ψ by n . (For any FL -module X one defines X^n to be X with the twisted action of L , that is, $l^n(x) := nln^{-1} \cdot x$, where $l \in L$, $x \in X$, $n \in N_L$.) Denote by $|N\Psi|$ the size of the N -orbit of Ψ . Then our generalization of Ballard's theorem asserts that $c_\Phi \geq |N\Psi| \cdot c_\Psi$ contains every PIM Ψ^n ($n \in N_L$) (Proposition 4.7).

2.3. Harish-Chandra induction. Let $G \in \{SL(n, q), n > 4, E_6(q), E_7(q), E_8(q)\}$, and let r be the rank of G . Let χ be the character of a PIM $\Phi \neq St$. We show (Section 6) that the Harish-Chandra theory together with the main result of Malle-Weigel [34] yields a lower bound $c_\Phi \geq r$. Here is the idea of the proof. In notation of Section 2.2, by analysis of the action of N on T (Proposition 4.12) and using Ballard's bound, we deduce that $c_\Phi \geq |N\beta| > r$, whenever $\beta \neq 1_T$. If $1_T = \beta$ then Ballard's bound is vacuous. In this case we first show that $c_\Phi = (\chi, 1_B^G)$, see Proposition 6.2. Let $\lambda \in \text{Irr } G$ be a common constituent of χ and 1_B^G . The Harish-Chandra theory tells us that $(\lambda, 1_B^G) \geq r$ for the above groups, unless $\lambda \in \{St, 1_G\}$. As St is not a constituent of χ , $c_\Phi < r$ implies $\lambda = 1_G$. So $c_\Phi = (\chi, 1_B^G) = (\chi, 1_G)$. By general modular representation theory, $(\chi, 1_G) \neq 0$ implies $\Phi = \Phi_1$ and $(\chi, 1_G) = 1$, and hence $c_\Phi = 1$. The groups G for which $c_\Phi = 1$ have been determined in [34].

We expect that this reasoning can be improved to obtain a lower bound for all classical groups, however, this requires much deeper analysis.

3. PRELIMINARIES

Let G be a finite group of order $|G|$ and p a prime number. Let ε be a primitive $|G|$ -root of unity. Any ordinary representation is equivalent to a representation ϕ over $\mathbb{Q}(\varepsilon)$, and moreover, over a maximal subring R of $\mathbb{Q}(\varepsilon)$ not containing $1/p$. In addition, R has a unique maximal ideal I such that $F = R/I$ is a finite field of characteristic p . Note that F

contains a primitive m -root of unity, where $m = |G|/|G|_p$. For uniformity one can similarly define R in the algebraic closure of \mathbb{Q} , and then fix this R to deal with all finite groups. The mapping $R \rightarrow F$ yields also a surjective homomorphism of the group of roots of unity in R to the group of roots of unity in F , used to define Brauer characters.

Every ordinary representation is equivalent to a representation over R . So if $\phi(G) \subset GL(n, R)$ for some n , then the natural projection $GL(n, R) \rightarrow GL(n, F)$ yields a p -modular representation $\bar{\phi} : G \rightarrow GL(n, F)$ called the *reduction of ϕ modulo p* . It is well known that the composition factors of $\bar{\phi}$ remain irreducible under any field extension of F . This can be translated to the language of RG - and FG -modules, however, it requires to consider only RG -modules that are free as R -modules. So *all RG -modules below are assumed to be free as R -modules*.

If G is a p' -group then, by Dickson's theorem, the reduction yields a bijection between the isomorphism classes of RG - and FG -modules which makes identical the p -modular representation theory with the ordinary representation theory of G . If p divides $|G|$, this is not true anymore, however, there is a rather sophisticated replacement: the reduction modulo p yields a bijection between the isomorphism classes of projective RG - and projective FG -modules (Swan's theorem, see [11, Theorem 77.2]).

Let $M \neq 0$ be a projective FG -module. Then the corresponding projective RG -module is called *the lifting of M* , which we often denote by \tilde{M} . Obviously, $\dim M$ is equal to the R -rank of \tilde{M} . The latter is equal to the dimension of the KG -module obtained from M_R by the extension of the coefficient ring to the quotient field K of R , so we also write $\dim M$ for the rank of an R -module M .

For sake of convenience we record the following easy observation:

Lemma 3.1. (1) *Let $M = M_1 \oplus M_2$ be a direct sum of FG - or RG -modules. Then $C_M(G) = C_{M_1}(G) \oplus C_{M_2}(G)$.*

(2) *If H is a subgroup of G , M a projective FG -module with lifting L then $\dim C_M(H) = \dim C_L(H)$.*

Proof. (1) is trivial. As the restriction of a projective G -module to H remains projective, it suffices to prove (2) for $H = G$. Then (2) is obvious if M is the regular FG -module. By (1), this implies (2) when M is free, and hence when M is projective. \square

The following is well known (see for instance [19, p. 52]):

Lemma 3.2. *Let $G = H \times N$, the direct product of finite groups H and N , and let Φ, Ψ be PIM's for H, N respectively. Then $\Phi \otimes \Psi$ is a PIM for G , and hence $c_{\Phi \otimes \Psi} = c_{\Phi} \cdot c_{\Psi}$.*

The following lemma asserts that, for a projective G -module M and a normal subgroup N of G , the G/N -module $C_M(N)$ is projective. This is a rather general fact, but it does not seem to be recorded in any standard textbook. The proof below is a variation of that given in [34, the proof of Proposition 2.2].

Lemma 3.3. *Let H be a subgroup of G and N a normal subgroup of H . Let M be a projective FG -module.*

(1) *$\overline{M}_{H/N}$ is a projective $F(H/N)$ -module.*

(2) Let L be the lifting of M . Then $\overline{L}_{H/N}$ is the lifting of $\overline{M}_{H/N}$.

(3) Let χ be the character of L . Then the truncation $\overline{\chi}_{H/N}$ is the character of $\overline{L}_{H/N}$.

Proof. (1) As $M|_H$ is a projective module, it suffices to prove the statement for $H = G$.

Assuming $H = G$, suppose first that M is the regular FG -module. Then $M|_N$ is a free FN -module of rank $|H : N|$, and hence $\dim \overline{FH} = |H : N|$. Let $a := \sum_{n \in N} n \in FN$. Then the mapping $h : x \rightarrow xa$ ($x \in FG$) is an FH -module homomorphism whose kernel A is spanned by the elements $g(n-1)$ ($n \in N, g \in H$). Therefore, $h(FH) = FH/A \cong F(H/N)$. As $xa \in \overline{FH}$ and $\dim F(H/N) = |H : N| = \dim \overline{FH}$, it follows that \overline{FH} is isomorphic to $F(H/N)$, the regular $F(H/N)$ -module.

Therefore, the lemma is true if M is free (and in this case \overline{M} is free). Otherwise, let M' be a projective FG -module such that $M \oplus M' = J$, where J is free. Then \overline{J} is a free $F(G/N)$ -module, and $\overline{M} \oplus \overline{M}' = \overline{J}$. As \overline{J} is a free $F(G/N)$ -module, \overline{M} is projective.

(2) Let $\pi : L \rightarrow M$ be the reduction map. Then $\pi(\overline{L}) \subseteq \overline{M}$. By Lemma 3.1, $\dim \overline{L} = \dim \overline{M}$, as desired.

(3) follows from the definition of $\overline{\chi}_{H/N}$. \square

For a p -group P , every projective FP -module M is free, that is, a direct sum of copies of the regular FP -module, which is the only PIM for P (see [11, §54, Exercise 1] or [20, Ch.III, Corollaries 2.9, 2.10]). This is therefore true for its lifting as well. It follows that both M and its lifting have the same rank as H -modules, equal exactly to $\dim M/|P|$. Obviously, $\dim C_M(P) = 1$ for the regular FP -module M . This implies the following assertion:

Lemma 3.4. *Let P be a p -subgroup of G , M a projective FG -module and $L = M_K$. Then $\dim C_M(P) = (\dim M)/|P| = (\dim L)/|P| = \dim C_L(P)$.*

Let $U \in \text{Syl}_p(G)$ and M a projective FG -module. Then $c_M := (\dim M)/|U|$ is an integer. Some formulas become simpler if one uses c_M instead of $\dim M$. The first equality of Lemma 3.4 implies:

Lemma 3.5. *Let H be a subgroup of G with $(|G : H|, p) = 1$. Then $c_M = c_{M|_H}$ for any projective FG -module M .*

Let M be a projective FG -module with lifting L . For brevity, the character χ of L is also called the character of M . It follows that χ vanishes at the p -singular elements (indeed, if $g = su$, where $u \neq 1$ is a p -element, s is a p' -element and $[s, u] = 1$, then L is a direct sum of the eigenspaces of s ; by the Krull-Schmidt theorem, every s -eigenspace is a free $R\langle u \rangle$ -module (as so is L), and hence the trace of su is 0). See [20, Ch.IV, Corollary 2.5].

Lemma 3.6. *Let G be a finite group and $U \in \text{Syl}_p(G)$. Let M be a projective FG -module with lifting L , and χ the character of L . Then $c_M = (\chi|_U, 1_U)$. Moreover, $C_M(U)$ and $C_L(U)$ (the fixed point subspaces of U on L, M , resp.) are $N_G(U)$ -modules with the same Brauer character.*

Proof. As $N_G(U)/U$ is a p' -group, $N_G(U)$ splits as UH , where $H \cong N_G(U)/U$. Therefore, $M|_H$ and $L|_H$ have the same Brauer characters. Let $\rho : L \rightarrow M$ be the reduction homomorphism. Obviously, $\rho(C_L(U)) \subset C_M(U)$. As $\dim C_L(U) = c_M = \dim C_M(U)$, we

have $\rho(C_L(U)) = C_M(U)$. It follows that $C_M(U)|_H$ and $C_L(U)|_H$ have the same Brauer characters, and the lemma follows. \square

Lemma 3.7. *Let N be a normal subgroup of G with $(|G : N|, p) = 1$. Let M be an FG -module.*

(1) $\text{Soc}(M|_N) = (\text{Soc } M)|_N$.

(2) *Let M be a PIM for G and let $S = \text{Soc } M$. Suppose that $S|_N$ is irreducible. Then $M|_N$ is a PIM for N .*

Proof. (1) Let X be an irreducible submodule of $M|_N$. Then so is gX for every $g \in G$, and hence $Y = \sum_{g \in G} gX$ is an FG -module, obviously, completely reducible. Therefore, $Y \subseteq \text{Soc } M$. As $\text{Soc}(M|_N)$ is the sum of irreducible submodules of $M|_N$, by the above we have $\text{Soc}(M|_N) \subseteq (\text{Soc } M)|_N$. The converse inclusion follows from Clifford's theorem.

(2) By (1), $\text{Soc}(M|_H)$ is irreducible, so $M|_H$ is a PIM (as it is projective). \square

The following lemma is a special case of [20, Ch.IV, Lemma 4.26]. Our proof below is different, and somehow, more natural.

Lemma 3.8. *Let N be a normal p -subgroup of G and let Φ be a PIM for G . Then $\overline{\Phi} := C_{\Phi}(N) = \text{Soc}(\Phi|_N)$ is a PIM for G/N and $c_{\Phi} = c_{\overline{\Phi}}$.*

Proof. By Lemma 3.3, $\overline{\Phi}$ is a projective $F(G/N)$ -module. As N acts trivially on every irreducible FG -module, $\text{Soc } \Phi \subseteq \overline{\Phi}$. In fact, $\text{Soc } \Phi = \text{Soc } \overline{\Phi}$ since G acts in $\overline{\Phi}$ via G/N . Recall that a projective module is a PIM if and only if its socle is irreducible. Therefore, $\text{Soc } \Phi$ is irreducible as an FG -module, and hence as an $F(G/N)$ -module. So $\overline{\Phi}$ is a PIM for G/N , as claimed.

Let $U \in \text{Syl}_p(G)$ and $\overline{U} = U/N$. Obviously, $C_{\Phi}(U) = C_{\overline{\Phi}}(\overline{U})$, so $c_M = \dim C_{\Phi}(U) = \dim C_{\overline{\Phi}}(\overline{U}) = c_{\overline{\Phi}}$, where $\overline{\Phi}$ is viewed as $F(G/U)$ -module. \square

Lemma 3.9. *Let H be a subgroup of G and M a projective FG -module. Then $M|_H$ is a projective module, and $c_M = \frac{|H|_p}{|G|_p} \cdot c_{M|_H}$.*

Proof. The first claim is well known. The second one follows by dividing by $|G|_p$ the left and the right hand sides of the equality $|G|_p \cdot c_M = \dim M|_H = c_{M|_H} \cdot |H|_p$. \square

Lemma 3.10. *Let $H \subset G$ be finite groups such that H contains a Sylow p -subgroup of G , and $N = O_p(H)$. Let M be a projective FG -module with socle S . Let $D = \text{Soc } \overline{S}_{H/N}$ and let M' be a projective $F(H/N)$ -module with socle D . Then $\overline{M}_{H/N}$ contains a submodule isomorphic to M' . In addition, $c_M \geq c_{M'}$.*

Proof. As $S \subset M$ and N is a p -group, we have $\text{Soc}(S|_H) \subseteq \text{Soc}(M|_H) \subseteq C_M(N)$. Viewing each module as an $F(H/N)$ -module, we have $D = \text{Soc}(\overline{S}_{H/N}) \subseteq \text{Soc}(\overline{M}_{H/N}) \subseteq \overline{M}_{H/N}$, and $\overline{M}_{H/N}$ is projective by Lemma 3.3. Therefore, $M' \subseteq \overline{M}_{H/N}$. The additional claim follows from Lemma 3.9, as $|H|_p = |G|_p$. \square

Lemma 3.11. *Let $H \subset G$ be finite groups. Suppose that $(|G : H|, p) = 1$ and Φ is a PIM of dimension $|G|_p$, that is, $c_{\Phi} = 1$. Then $\Psi := \Phi|_H$ is a PIM for H , and $c_{\Psi} = 1$. In addition, $\text{Soc}(\Phi|_H)$ is irreducible.*

Proof. This follows straightforwardly from Lemma 3.9. \square

Lemma 3.12. *Let G be a finite group and U a Sylow p -subgroup of G . Let $\eta : G \rightarrow \mathbb{C}$ be a conjugacy class function such that $\eta(g) = 0$ for every $1 \neq g \in U$. Let τ be an irreducible character of U . Then $(\eta, \tau^G) = \eta(1)\tau(1)/|U|$. In particular, if η is a character of a PIM Φ then $(\eta, \tau^G) = \tau(1) \cdot c_\Phi$.*

Proof. By Frobenius reciprocity we have: $(\eta, \tau^G) = (\eta|_U, \tau) = \eta(1)\tau(1)/|U|$.

Corollary 3.13. *Let G be a Chevalley group in defining characteristic p , U a Sylow p -subgroup of G and let η be the character of a p -modular PIM Φ . Then $(\eta, 1_U^G) = c_\Phi = (\eta, \Gamma)$, where Γ denotes a Gelfand-Graev character of G .*

Note that every Gelfand-Graev character of G is induced from a certain one-dimensional character of U , see [16] or [15].

Lemma 3.14. *Let \mathbf{H} be a reductive algebraic group and Fr a Frobenius endomorphism of \mathbf{H} . Let \mathbf{G} be the semisimple part of \mathbf{H} and $H = \mathbf{H}^{Fr}$, $G = \mathbf{G}^{Fr}$. Then:*

(1) *Every irreducible FH -module M remains irreducible under restriction to G . Consequently, if Φ is a PIM for H then $\Phi|_G$ is a PIM for G .*

(2) *Let Ψ be a PIM for G with character η and let $\lambda \in \text{Irr } G$. For $h \in H$ denote by λ^h the h -twist of λ . Then $(\lambda, \eta) = (\lambda^h, \eta)$. In other words, the rows of the decomposition matrix of G corresponding to H -twisted ordinary characters coincide.*

Proof. (1) It is known that H is a group with BN-pair [33, 24.10]. Let B be a Borel subgroup of H and $U = O_p(H)$. By [8, Theorem 4.3(c)], $\dim C_M(U) = 1$. As H/G is a p' -group, $U \subset G$. By Clifford's theorem, $M|_G$ is completely reducible, and if $M|_G$ is reducible then $\dim C_M(U) > 1$, which is false.

The additional statement in (1) follows from Lemma 3.7.

(2) It follows from (1) that $\Psi = \Psi^h$. So $(\lambda, \eta) = (\lambda^h, \eta^h) = (\lambda^h, \eta)$. \square

Proposition 3.15. *Let $H \subset G$ be finite groups such that $(|G : H|, p) = 1$ and $N = O_p(H)$. Let M be a projective FG -module with socle S and character χ . Let $D \subseteq \text{Soc}(S|_H)$ be an irreducible $F(H/N)$ -module and let η be the Brauer character of D . Let $\lambda \in \text{Irr } H/N$. Suppose that η is a constituent of $\lambda \pmod{p}$ with multiplicity d . Then $(\chi, \lambda^{\#G}) \geq d$.*

Proof. Let R be the projective $F(H/N)$ -module with socle D and character ρ . In fact, R is a PIM as D is irreducible. Then $(\rho, \eta) = 1$ by the Brauer reciprocity [20, Lemma 3.3]. Furthermore, $(\overline{\chi}, \eta) \geq 1$ as $R \subseteq \overline{M}$. As $\lambda \pmod{p}$ contains η with multiplicity d , it follows that $(\overline{\chi}, \lambda) \geq d$. By the Harish-Chandra reciprocity [13, 70.1(iii)], we have $(\chi, \lambda^{\#G}) = (\overline{\chi}_{H/N}, \lambda) \geq d$. \square

Corollary 3.16. *Let G be a Chevalley group, and P be a parabolic subgroup. Let Φ be a PIM with character χ and socle S . Suppose that $\text{Soc}(S|_P)$ lifts, and let λ be the character of this lift. Then $(\chi, \lambda^G) > 0$.*

Proof. This is a special case of Proposition 3.15 with $M = \Phi$ and $H = P$. \square

4. LOWER BOUNDS FOR PIM DIMENSIONS

Let G be a quasi-simple Chevalley group, and let B, N be subgroups defining a BN -pair structure of G . Here B is a Borel subgroup of G , $U = O_p(B)$ and let T_0 be a maximal torus of B . Then $W_0 = N/T_0$ is the Weyl group of G as a group with BN -pair. If G is non-twisted then W_0 coincides with W , the Weyl group of \mathbf{G} .

Every irreducible character β of T_0 inflated to B yields a character of B , trivial on U , which we denote by β_B . Obviously, $\beta \rightarrow \beta_B$ is a bijection between $\text{Irr } T_0$ and the 1-dimensional characters of B trivial on U . Therefore, the induced representation β_B^G coincides with $\beta^{\#G}$.

Recall that a PIM Φ has an irreducible socle S ; so the socle of $\Phi|_B$ contains the socle of $S|_B$.

Lemma 4.1. *Let G be a Chevalley group viewed as a group with BN -pair (so $W_0 := W(T_0)$ is the Weyl group of the BN -pair). Let χ be a character of G vanishing at all unipotent elements $g \neq 1$. Then $c_\chi = (\chi, 1_U^G) = \sum_\beta |W_0\beta| \cdot (\chi, \beta_B^G) = \sum_\beta |W_0\beta| \cdot (\bar{\chi}_{T_0}, \beta)$, where β runs over representatives of the W_0 -orbits in $\text{Irr } T_0$. In particular, if χ is the character of a PIM Φ then $c_\Phi = \sum_\beta |W_0\beta| \cdot (\bar{\chi}_{T_0}, \beta)$.*

Proof. By Lemma 3.12, 3.13, $|U| \cdot (\chi, 1_U^G) = \chi(1)$. Note that $1_U^G = \oplus_{\beta_0 \in \text{Irr } T_0} \beta_B^G$. Let $\beta' \in \text{Irr } T$. By [40, Theorem 47], β_B^G and β'^G_B are equivalent if and only if β and β' are in the same W_0 -orbit. It follows that $1_U^G = \oplus_\beta |W_0| \beta_B^G$, where β runs over representatives of the W_0 -orbits in $\text{Irr } T_0$. This implies the first equality, while the second one follows from the Harish-Chandra reciprocity formula $(\chi, \beta_B^G) = (\bar{\chi}_{T_0}, \beta)$. If χ is the character of Φ then $c_\Phi = \chi(1)/|U|$. \square

Lemma 4.2. *Let S be a finite p -group with normal subgroup K . Let χ be a character vanishing at all non-identity p -elements of S . Then $\chi = c_\chi \cdot \rho_S^{\text{reg}}$ and $c_\chi = c_{\bar{\chi}_{S/K}}$ (where ρ_S^{reg} denotes the regular character of S).*

Proof. Let τ be an irreducible character of S . Then $(\chi, \tau) = \tau(1)\chi(1)/|S| = c_\chi \cdot \tau(1)$, whereas $(\rho_S^{\text{reg}}, \tau) = \tau(1)$. So the former claim follows. Let M be the $\mathbb{C}S$ -module with character χ . It follows that M is a free $\mathbb{C}S$ -module of rank c_χ . The latter claim is obvious for the regular $\mathbb{C}S$ -module in place of M , which implies the lemma. \square

Lemma 4.3. *Let H be a finite group and $U = O_p(H)$. Let χ be a character of G vanishing at all non-identity p -elements of G . Then $\bar{\chi}_{H/U}$ vanishes at all non-identity p -elements of H/U and $c_\chi = c_{\bar{\chi}_{H/U}}$.*

Proof. Let M be a $\mathbb{C}H$ -module with character χ , and let M' be the fixed point subspace of U on M . Note that $x \in M'$ if and only if $x = \frac{1}{|U|} \sum_{u \in U} um$ for some $m \in M$. Let $g \in H$, $u \in U$. Suppose that the projection of g , and hence of gu , into H/U is not a p' -element (the projections of gu and g in H/U coincide). It follows that $\bar{\chi}_{H/U}(g) = \frac{1}{|U|} \sum_{u \in U} \chi(gu) = 0$ by assumption, whence the first claim. The equality $c_\chi = c_{\bar{\chi}_{H/U}}$ follows from Lemma 4.2. \square

Proposition 4.4. *Let $G, B = UT_0, W_0$ be as in Proposition 4.1, and let χ be a character of G vanishing at all p -elements $1 \neq u \in G$. Let $\beta \in \text{Irr } T_0$ and β_B the inflation of β to*

B. Suppose that β is an irreducible constituent of $\overline{\chi}_{B/U}$. Then $c_\chi \geq |W_0\beta|$. In addition, if $\chi(1) = |G|_p$ then $\overline{\chi}_{B/U}$ is irreducible and W_0 -invariant.

Proof. By [40, Theorem 47], β_B^G is equivalent to $w(\beta)_B^G$ for every $w \in W_0$. By the Frobenius reciprocity, $(\chi|_B, \beta_B) = (\chi, \beta_B^G) = (\chi, w(\beta)_B^G) = (\chi|_B, w(\beta)_B)$. Therefore, both β_B and $w(\beta)_B$ occur in $\chi|_B$ with equal multiplicity. As $w(\beta)_B$ is trivial on $U = O_p(B)$, it follows that $w(\beta)$ is a constituent of $\overline{\chi} := \overline{\chi}_{B/U}$. So $\overline{\chi}(1) \geq |W_0\beta|$. As B/U is a p' -group, $c_{\overline{\chi}} = \overline{\chi}(1)$. We know that $c_\chi \geq c_{\overline{\chi}}$ (Lemma 4.2). So the result follows. This also implies the additional statement, as $1 = c_\chi \geq c_{\overline{\chi}} \geq |W_0\beta|$ means that β is W_0 -stable. \square

Proposition 4.5. *Let $G, B = UT_0, W_0$ be as in Proposition 4.1. Let Φ be a PIM with socle S and character χ , and let β_B be the Brauer character of $\text{Soc } S|_B$. Then $c_\Phi \geq |W_0\beta|$.*

Proof. Let χ be the character of Φ , that is, the character of the lifting M of Φ . By Lemma 3.6, $c_\Phi = (\chi|_U, 1_U) = c_\chi$, and the character of $C_M(U)|_B$ coincides with the Brauer character of $C_\Phi(U)|_B$. Therefore, β_B occurs as a constituent of $C_M(U)|_B$ so $(\chi|_B, \beta_B) \geq 1$, and hence β is a constituent of $\overline{\chi}_{B/U}$. So the result follows from Proposition 4.4. \square

Remarks. (1) Let \mathbf{G} be the algebraic group defining G as $G = \mathbf{G}^{Fr}$. Then in Proposition 4.5 $S = V_\mu|_G$ for some irreducible \mathbf{G} -module V_μ , where μ is the highest weight of V_μ . Moreover, $\beta = \mu|_{T_0}$. If G is a non-twisted Chevalley group then $W(T_0) = W(G)$. Therefore, for non-twisted groups the result coincides with that of Ballard [1, Corollary 5.4], see also [27, 9.7]. (2) Recall that β_B is irreducible, whereas $\overline{\chi}_{B/U}$ may be reducible.

This can be generalized to a parabolic subgroup P in place of a Borel subgroup B , and a Levi subgroup L of P in place of T_0 . However, the statement has to be modified. For this we need to replace W_0 by a certain group \overline{W}_L , which is contained in $N_G(L)/L$. Specifically, we may assume that $B \subseteq P$, and that $T_0 \subseteq L$. (The equality holds only if $P = B$.) Using the data B, N, W_0 , defining the BN -pair structure of G , we set $N_L = \{n \in N : nLn^{-1} = L\}$. Then $\overline{W}_L = N_L/(N_L \cap L)$. (Note that \overline{W}_L is not the Weyl group of L viewed as a group with BN -pair; the latter is $(N_L \cap L)/T^0$.) For a character $\lambda \in \text{Irr } L$ and $n \in N_L$ one considers the n -conjugate λ^n of λ . Of course, $\lambda^n = \lambda$ if $n \in L$. Therefore, λ^n depends only on the coset $w := n \cdot (N \cap L)$, which is an element of \overline{W}_L . So one usually writes λ^w for $w \in \overline{W}_L$, with the meaning that $\lambda^w = \lambda^n$ for n from the pullback of w in N_L . If $L = B$ then \overline{W}_L is exactly W_0 .

Recall (see Notation) that $\overline{\chi}_L$ denotes the Harish-Chandra restriction (or the truncation) of χ , and $\overline{\chi}_L$ coincides with $\chi'|_L$, where χ' is a character of P trivial on $O_p(P)$ and such that $\chi|_P = \chi' + \mu$ for some character μ whose all irreducible constituents are non-trivial on $O_p(P)$.

Lemma 4.6. *Let P be a parabolic subgroup of G and L a Levi subgroup of P . Let χ be a character of G . Then $\overline{\chi}_L$ is \overline{W}_L -invariant. In particular, if $P = B$ then $\overline{\chi}_{T_0}$ is W_0 -invariant.*

Proof. Let λ' be an irreducible constituent of χ' , and $\lambda = \lambda'|_L$. By the Frobenius reciprocity, $(\chi, \lambda'^G) = (\chi|_P, \lambda') = (\chi', \lambda')$ as λ' is trivial on $O_p(P)$. Then (χ', λ') equals

$(\chi'|_L, \lambda) = (\bar{\chi}_L, \lambda)$. Furthermore, $\lambda'^G = w(\lambda')^{\#G}$ for every $w \in \bar{W}_L$, see [13, 70.11]. Hence $(\chi', \lambda) = (\chi', w(\lambda))$, and the result follows. \square

The group N_L acts on L by conjugation, and hence \bar{W}_L acts on $\text{Irr } L$. Note that for any finite group G the correspondence $\Phi \rightarrow \text{Soc } \Phi$ is compatible with the automorphism group action. In other words, if h is an automorphism of G and Φ^h is the h -twist of Φ , then $\text{Soc } \Phi^h = (\text{Soc } \Phi)^h$.

Proposition 4.7. *Let G be a Chevalley group, and let P be a parabolic subgroup of G with Levi subgroup L . Let Φ be a PIM with socle S . Let $S_1 = \text{Soc}(S|_P)$ and $S_L = S_1|_L$. Let Ψ be the projective FL -module with socle S_L . Then $c_\Phi \geq |\bar{W}_L : C_{\bar{W}_L}(S_L)| \cdot c_\Psi$.*

Proof. Let M be the lifting of Φ . Then \bar{M}_L and $\bar{\Phi}_L$ are projective L -modules with the same character $\bar{\chi}_L$, see Lemma 3.10. (By convention we call $\bar{\chi}_L$ the character of $\bar{\Phi}_L$.) Note that $O_p(L) = 1$, so $\bar{\chi}_L$ coincides with the character in Lemma 4.6, which tells us that $\bar{\chi}_L$ is \bar{W}_L -invariant. As a projective FL - and RL -module is determined by its character, it follows that $\bar{M}_L^w = \bar{M}_L$ and $\bar{\Phi}_L^w = \bar{\Phi}_L$ for every $w \in \bar{W}_L$.

By Lemma 2.1, S_L is irreducible. As $O_p(P)$ acts trivially on S_L , it follows that $S_L \subseteq \text{Soc } \bar{\Phi}_L$. Then $\Psi \subseteq \bar{\Phi}_L$. By the comment prior to the proposition, the \bar{W}_L -orbits of S_L and Ψ are of the same size $l := |\bar{W}_L : C_{\bar{W}_L}(S_L)|$. As $\bar{\Phi}_L$ is \bar{W}_L -invariant, every Ψ^w ($w \in \bar{W}_L$) is in $\bar{\Phi}_L$. Therefore, $\bar{\Phi}_L$ contains at least l distinct PIM's $\bar{\Psi}^w$. Obviously, $\dim \Psi^w = \dim \Psi$ for $w \in \bar{W}_L$, so $\dim \bar{\Phi}_L \geq l \cdot \dim \Psi$, and hence $c_{\bar{\Phi}_L} \geq l \cdot c_\Psi$. By Lemma 3.10, $c_\Phi \geq c_{\bar{\Phi}_L} \geq l \cdot c_\Psi$, as required. \square

In the remaining part of this section we discuss the question when the lower bounds provided in Proposition 4.5 and Proposition 4.7 are efficient. It is known from Ballard's paper [1] that the bound is sharp for some PIM's, and more examples are provided in [27, §10.7]. However, in general the bound is not sharp, and, especially for twisted groups, there are some characters $1_{T_0} \neq \beta \in \text{Irr } T_0$ for which the bound is too small for efficient use. The situation is better for some non-twisted Chevalley groups; this will be explained in the rest of this section.

If $C_{W_0}(\beta) = W_0$, then Proposition 4.5 gives $c_\Phi \geq 1$, which is trivial. This always happens if G is a non-twisted Chevalley group $G(q)$ with $q = 2$, or, in general, if $\beta = 1_{T_0}$. In fact, there are more cases where $C_{W_0}(\beta) = W_0$. In addition, one needs to decide what is the minimum size of $|W_0\beta|$ if it is greater than 1. Thus, we are faced with two problems:

- (1) Determine $\beta \in \text{Irr } T_0$ such that $C_{W_0}(\beta) = W_0$, and
- (2) Assuming $|W_0\beta| > 1$, find a lower bound for $|W_0\beta|$.

We could obtain a full solution to these problems. However, it seems that for the purpose of this paper we need only to describe favourable situations, where the orbit $W_0\beta$ is not too small for every $\beta \neq 1_{T_0}$. Our results in this line are exposed in Propositions 4.9, 4.11 and 4.12, where the groups considered are non-twisted. To explain our approach, we therefore assume that G is non-twisted. In this case W_0 coincides with the Weyl group W of \mathbf{G} .

Let \mathbf{G} be a simple simply connected algebraic group in defining characteristic p , and $G = G(q)$. Let r be the rank of \mathbf{G} and let \mathbf{T}_0 be a maximal torus of \mathbf{G} . Then the action of $W = N_{\mathbf{G}}(\mathbf{T}_0)$ on \mathbf{T}_0 yields the action of W on \mathbb{Z}^r , the group of rational characters of

T₀. In turn, this yields a representation $\zeta_0 : W \rightarrow GL(r, \mathbb{Z})$, which we call the natural representation of W . It is well known that $\zeta_0(W)$ is an irreducible group (of $GL(n, \mathbb{C})$) generated by reflections. (See [15, 0.31].) The representation ζ_0 is well understood, see [41, 40]. It turns out that, if $\beta \neq 1_{T_0}$, then $|W\beta|$ is not too small provided $\zeta_0(W)$ remains irreducible modulo every prime dividing $q - 1$. This requires q to be even for G of type B, C, D , see Table 3. For a prime ℓ dividing $q - 1$ denote by ζ_ℓ the representation obtained from ζ_0 by reduction modulo ℓ . More precise analysis shows that it is enough that the dual representation of ζ_ℓ , if it is reducible, were fixed point free. This happens for $SL(r + 1, q)$, $r > 1$, when ℓ divides $r + 1$, and for $E_6(q), E_7(q), E_8(q)$, see Proposition 4.11.

Lemma 4.8. *Let $G = G(q)$ be a non-twisted Chevalley group of rank n , T_0 a split torus, and W the Weyl group of G . Let $\zeta_0 : W \rightarrow GL(n, \mathbb{Z})$ be the natural representation of W and let ζ_ℓ denote the reduction of ζ_0 modulo a prime ℓ . Then T_0 has a non-trivial W -invariant character if and only if there is a prime ℓ dividing $q - 1$ such that ζ_ℓ fixes a non-zero vector on F_ℓ^n . In addition, ζ_ℓ is dual to the natural action of W on T_ℓ , the subgroup of elements of order ℓ in T_0 .*

Proof. If W fixes a character $1_{T_0} \neq \beta \in \text{Irr } T_0$ then it fixes any power β^k too. So it suffices to deal with the case where the order of β is a prime. So let $\ell = |\beta|$ be a prime.

It is well known that T_0 is a direct product of cyclic groups of order $q - 1$, and hence $T_\ell = \{t^{(q-1)/\ell} : t \in T_0\}$. The characters of T_0 therefore correspond to elements of $\mathbb{Z}^n / (q - 1)\mathbb{Z}^n$, and those of T_ℓ correspond to elements of $\mathbb{Z}^n / \ell\mathbb{Z}^n$. This yields the reduction mapping $\zeta_0 \rightarrow \zeta_\ell$, and the first assertion of the lemma follows.

The additional claim describes ζ_ℓ in terms of the action of W on T_ℓ . The group $T_0^* := \text{Irr } T_0$ is isomorphic to T_0 , and the actions of W on T_0 and on T_0^* are dual to each other. Let $T_\ell^* = \{t \in T_0^* : t^\ell = 1\}$. Then the action of W on T_ℓ^* is dual to the action of W on T/T^ℓ . As T is homocyclic, the action of W on T/T^ℓ is equivalent to that on T_ℓ . \square

Proposition 4.9. *Let q be even, $G = C_n(q)$, $n > 1$, or $D_n^+(q)$, $n > 3$, and let $1_{T_0} \neq \beta \in \text{Irr } T_0$. Then $|W_0\beta| \geq 2n$.*

Proof. The group $W \cong W(B_n) = W(C_n)$, resp., $W(D_n)$, is a semidirect product of a normal 2-group A of order 2^n , resp., 2^{n-1} , and the symmetric group S_n . Note that $|A| \geq 2n$. In the reflection representation ζ_0 the group $\zeta_0(W)$ can be realized by monomial $(n \times n)$ -matrices over \mathbb{Z} with diagonal subgroup A and the group S_n as the group of all basis permuting matrices. This group remains irreducible under reduction modulo any prime $\ell > 2$. In addition, A fixes no non-zero vector on F_ℓ^n , and if $G = D_n^+(q)$ then $\det a = 1$ for $a \in A$.

It suffices to prove the lemma when $|\beta| = \ell$ for every prime divisor ℓ of $|T_0|$. As $|T_0|$ is odd, ℓ is odd too. So ζ_ℓ , the reduction of ζ_0 modulo ℓ , is an irreducible matrix group. Let $0 \neq v \in F_\ell^n$. Set $X = C_W(v)$ and $Y = A \cap X$. Since A acts fixed point freely on \mathbb{Z}^n , and hence on F_ℓ^n , it follows that $Y \neq A$. We show that $|W : X| \geq 2n$. If $Y = 1$ then $|W : X| \geq |A| \geq 2n$. Suppose $Y \neq 1$. If $X = Y$ then $|W : X| \geq 2|S_n| \geq 2n$. Suppose $X \neq Y$, and let $S = X/Y$. Then $S \subset S_n$. As $|A : Y| \geq 2$, we have $|W : X| = |A : Y| \cdot |S_n : S| \geq 2 \cdot |S_n : S|$. If $|A : Y| = 2$ then v is a basis vector, and $S \cong S_{n-1}$. Therefore, $|W : X| = 2n$. Suppose that $|A : Y| > 2$. It is easy to check that this implies that $|Y| = 2^i$ with $i > 1$ and $X \cong S_i \times S_{n-i}$. This again implies $|W : X| \geq 2n$, as required. \square

In Tables 1, 2, 3 R is an indecomposable root system, and Z_2 in Table 2 denotes the cyclic group of order 2. Note that the data in Tables 2,3 are well known for root systems of types A, B, C, D , and for types E_i , $i = 6, 7, 8$, the data follow from [7, 28].

TABLE 1: The structure of $W(R)$

R	A_{n-1}	B_n, C_n	D_n	E_6	E_7	E_8	F_4	G_2
W	S_n	$2^n \cdot S_n$	$2^{n-1} \cdot S_n$	$PSp(4, 3) \cdot Z_2$	$Z_2 \cdot Sp(6, 2) \cdot Z_2$	$\Omega^+(8, 2) \cdot Z_2$	$\Omega^+(4, 3) \cdot Z_2$	$Z_2 \times S_3$

TABLE 2: The minimum degree of a non-linear irreducible representation of $W(R)$

R	$A_{n-1}, B_n, C_n, D_n, n \neq 4$	$A_3, B_4 = C_4, D_4$	E_6	E_7	E_8	F_4, G_2
d	$n - 1$	2	6	7	8	2

TABLE 3: mod ℓ irreducibility of the natural representation of $W(R)$

R	A_{n-1}	B_n, C_n	D_n	E_6	E_7	E_8	F_4	G_2
ℓ	$(\ell, n) \neq 1$	$\ell \neq 2$	$\ell \neq 2$	$\ell \neq 3$	$\ell \neq 2$	any ℓ	$\ell \neq 2$	$\ell \neq 3$
dim	$n - 1$	n	n	6	7	8	4	2

Lemma 4.10. *Let d be the minimum degree of a non-linear irreducible character of $W(R)$. Then d is as in Table 2.*

Proof. If $R = A_{n-1}$ then $W(R) \cong S_n$. The degree formula for irreducible representations of S_n easily implies that $d \geq n - 1$ unless $n = 4$, where $d = 2$. Let $R = B_n, C_n$ or D_n . It is well known that the minimum degree of a faithful representation of $W(R)$ equals $2n$. As $W(R)/A \cong S_n$ for an abelian normal subgroup A , one arrives at the same conclusion as for S_n . If $R = F_4$ then $W(R) \cong O^+(4, 3)$. This group has a normal series $N_1 \subset N_2 \subset W(R)$, where N_1 is extraspecial 2-subgroup of order 32, N_2/N_1 is elementary abelian of order 9, and $W(R)/N_2$ is elementary abelian of order 4. One observes that $W(R)/N_1$ is isomorphic to $S_3 \times S_3$, and this group has an irreducible character of degree 2. Groups $Sp(4, 3)$, $Sp(6, 2)$ and $\Omega^+(8, 2)$ are available in [7], so the result follows by inspection. \square

Below $G = E_6(q), E_7(q)$ are groups arising from the simply connected algebraic group.

Proposition 4.11. *Let $G \in \{SL(n, q), n > 2, E_6(q), E_7(q), E_8(q), F_4(q), G_2(q)\}$. Let W be the Weyl group of G , and T_0 a split torus. Then 1_{T_0} is the only W -invariant irreducible character of T_0 .*

Proof. We use Lemma 4.8 without explicit references to it.

Case 1. $G = SL(n, q)$, $n > 2$. Here $W \cong S_n$. We can assume that T_0 is the group of diagonal matrices. Let ζ_0 be the usual irreducible representation of $S_n \rightarrow GL(n - 1, \mathbb{Z})$. Let ℓ be a prime dividing $|T_0|$. If $(\ell, n) = 1$ then ζ_0 remains irreducible modulo ℓ . So assume $(\ell, n) \neq 1$. Then ζ_ℓ is not completely reducible. It has two composition factors, one is of dimension $n - 2$, and the other factor is trivial (see [30, 5.3.4]). Let η be a non-trivial ℓ -root of unity in F_q and $t = \eta \cdot \text{Id}$; then t is a scalar matrix. Obviously, $t \in T_0$. It follows that

$\zeta_\ell^*(W)$ fixes a non-zero vector of F_ℓ^{n-1} . As ζ_ℓ^* is dual to ζ_ℓ , it follows by dimension reason that ζ_ℓ has no fixed vector, unless, possibly, $n-1=2$. If $n=3$ then $\ell=3$. As ζ_3^* is faithful and reducible, $\zeta_3^*(W)$ is conjugate with the matrix group $\left\{ \begin{pmatrix} 1 & * \\ 0 & \pm 1 \end{pmatrix} \right\}$. Then $\zeta_3(W)$ fixes no non-zero vector on F_3^2 .

Case 2. $G = E_6(q)$. Here $\zeta_0(W) \subset GL(6, \mathbb{Z})$ and $W \cong PSp(4, 3) \cdot Z_2$. Then ζ_0 remains irreducible modulo any prime $\ell \neq 3$, see [28]. Let $\ell = 3$, so 3 divides $q-1$ and $|Z(G)| = (3, q-1) = 3$. Therefore, W fixes a non-identity element of T_3 . So $T_3 \cong F_3^6$ viewed as a W -module has a one-dimensional subspace S , say, fixed by W . As $PSp(4, 3)$ has no non-trivial irreducible representation of degree less than 5 [28], it follows that the second composition factor is of degree 5. As T_3 is indecomposable, and the quotient T_3/S is irreducible, the dual module T_3^* has no trivial submodule.

Case 3. $G = E_7(q)$. Here $\zeta_0(W) \subset GL(7, \mathbb{Z})$ and $W \cong Z_2 \cdot Sp(6, 2) \cdot Z_2$. Then ζ_0 remains irreducible modulo any prime $\ell > 2$. Let $\ell = 2$. Then q is odd and $|Z(E_7(q))| = (2, q-1) = 2$. Therefore, the module T_2 has a non-trivial fixed point submodule S . As T_2 is indecomposable, and the quotient T_2/S is irreducible, the dual module T_2^* has no trivial submodule.

Case 4. $G = E_8(q)$. Here $\zeta_0(W) \subset GL(8, \mathbb{Z})$ and $W \cong Z_2 \cdot \Omega^+(6, 2) \cdot Z_2$. Then ζ_0 remains irreducible modulo any prime ℓ , see [28].

Case 5. $G = F_4(q)$. Then $W \cong W(F_4) \cong O^+(4, 3)$ and $\zeta_\ell(W)$ is irreducible for $\ell > 2$, see Table 2. Note that $O^+(4, 3)' \cong SL(2, 3) \circ SL(2, 3)$. So it suffices to observe that $O^+(4, 3) \bmod 2$ fixes no non-zero vector on F_2^4 . As this is the case for the Sylow 3-subgroup of W , the result follows.

Case 6. $G = G_2(q)$. Then W is the dihedral group of order 12. Then $\zeta_\ell(W)$ is irreducible modulo any prime $\ell \neq 3$. Let $\ell = 3$. Then a Sylow 2-subgroup of W fixes no non-zero vector in F_3^2 , and hence this is true for the dual action. So again W fixes no element of order 2 of $\text{Irr } T_0$. \square

Proposition 4.12. *Let $G \in \{SL(n, q), n > 4, E_6(q), E_7(q), E_8(q)\}$. Let $1_{T_0} \neq \beta \in \text{Irr } T_0$. Then $|W\beta| \geq m$, where $m = n, 27, 28, 120$, respectively.*

Proof. By Proposition 4.11, $|W\beta| > 1$. Let r be the rank of G . If W is realized via ζ_0 as a subgroup of $GL(r, \mathbb{Z})$ generated by reflections then the stabilizer $C_W(v)$ of every vector $v \in \mathbb{Z}^r$ is generated by reflections. Due to a result of J.-P. Serre, see [29], this is also true if W acts in F_ℓ^r and ℓ is coprime to $|W|$. This makes easy the computation of $|W\beta| = |W : C_W(v)|$. If ℓ divides $|W|$, and $v \in F_\ell^{r_0}$ then $C_W(v)$ is not always generated by reflections; see [29] where the authors classify all finite irreducible groups H such that $C_H(V)$ is generated by reflections for every subspace V of F_ℓ^r . Partially we could use the results of [29], but it looks simpler to argue in a more straightforward way. For our purpose, in most cases it suffices to know the index of a maximal non-normal subgroup of W , which can be read off from [7].

Let ℓ be a prime dividing $|\beta|$. As $|W : C_W(\beta)| \geq |W : C_W(\beta^k)|$, it is sufficient to deal with the case where $|\beta| = \ell$ is a prime. Let D be the derived subgroup of W . If $\zeta_\ell(D)$

is irreducible then $|W : C_W(\beta)| \geq |D : C_D(\beta)|$, which is not less than the index m_D of a maximal subgroup in D .

If $G = SL(n, q)$ then $W_0 \cong S_n$ and $D \cong \mathcal{A}_n$, the alternating group. It is well known that every proper subgroup of \mathcal{A}_n , $n > 4$, is of index at least n . So the lemma follows in this case. Let $G = E_6(q)$; then $D \cong SU_4(2)$ and $m_D = 27$, see [7]. Let $G = E_7(q)$; then $D \cong Sp_6(2)$ and $m_D = 28$ [7]. If $G = E_8(q)$ then $D/Z(D) \cong O_8^+(2)'$ and $m_D = 120$ [7]. \square

Remark. If $G = SL(n, q)$, q odd, $n = 4$ then there is an element $\beta \neq 1_{T_0}$ with $|W\beta| = 3$. Indeed, the Sylow 2-subgroup X of S_4 has index 3, and fixes a non-zero vector of F_2^3 . By Proposition 4.11, this vector is not fixed by S_4 , whence the claim. Let $G = SL(3, q)$. Then $W \cong S_3$, and ζ_ℓ is irreducible for every $\ell \neq 3$. In this case $m = 3$. Let $\ell = 3$. It is observed in the proof of Proposition 4.11 that $\zeta_3^*(W)$ fixes no non-zero vector. However, the Sylow 3-subgroup of W fixes a vector $v \neq 0$. Then $|\zeta_3^*(W)v| = 2$, so $m = 2$.

5. PARABOLIC DESCENT

Recall that for a PIM Φ of a group G we set $c_\Phi = \dim \Phi / |G|_p$. If H is a normal subgroup of G and M is an FG -module, then $C_M(H)$, the fixed point submodule for H , is viewed as $F(G/H)$ -module. The socle of a module M is denoted by $\text{Soc } M$. Every PIM is determined by its socle. The PIM whose socle is 1_G is called here 1-PIM, and denoted by Φ_1 .

Let G be a Chevalley group so (see Notation) $G = \mathbf{G}^{Fr}$, where \mathbf{G} is simple and simply connected. Let P be a parabolic subgroup of G with Levi subgroup L . The parabolic descent is the mapping $\pi_{G,P} : \Phi \rightarrow \Psi$, where Φ runs over the set of PIM's for G and Ψ is a PIM for L . (One can extend this to the \mathbb{Z} -lattices spanned by PIM's for G and L .) The parabolic descent $\pi_{G,P} : \Phi \rightarrow \Psi$ is determined by the Smith-Dipper mapping $\sigma_{G,L} : \text{Irr } G \rightarrow \text{Irr } L$ defined by $S \rightarrow \text{Soc}(S|_P)$, where $S \in \text{Irr } G$ and the right hand side is viewed as an FL -module.

Lemma 5.1. *Let P be a parabolic subgroup of G and L a Levi of P . The mappings $\sigma_{G,P} : \text{Irr } G \rightarrow \text{Irr } L$ and $\pi_{G,P} : \text{PIM}_G \rightarrow \text{PIM}_L$ are surjective.*

Proof. Let M be an irreducible FP -module trivial on $O_p(P)$. There exists an irreducible FG -module R such that $\text{Hom}(M^G, R) \neq 0$. By Frobenius reciprocity [12, 10.8], $\dim \text{Hom}(M^G, R) = \dim \text{Hom}(M, R|_P)$. So M is isomorphic to a submodule M' , say, of $R|_P$. So $M' \subseteq \text{Soc } R|_P$. By Lemma 2.1, $\text{Soc } R|_P$ is irreducible so $M' = \text{Soc } R|_P$. This implies the statement for $\sigma_{G,P}$. In turn, this implies the statement for $\pi_{G,P}$ as, both for G and L , irreducible modules are in natural bijection with PIM's. \square

Let L' be the subgroup of L generated by all unipotent elements of L . If Ψ is a PIM for L then $\Psi|_{L'}$ is a PIM for L' (Proposition 3.14). Then one may also consider the mapping $\pi_{G,L'}$ which sends Φ to $\Psi|_{L'}$. For our purpose this version of the parabolic descent has some advantage. Indeed, there are a parabolic subgroup \mathbf{P} of \mathbf{G} and a Levi subgroup \mathbf{L} of \mathbf{P} such that $P = \mathbf{P}^{Fr}$ and $L = \mathbf{L}^{Fr}$. Let \mathbf{L}' denote the semisimple part of \mathbf{L} . Then $L' = (\mathbf{L}')^{Fr}$. Thus, L' corresponds to a semisimple subgroup \mathbf{L}' of \mathbf{G} , and hence the irreducible representations of L' can be parameterized in terms of highest weights. This allows us to make more precise control of $\pi_{G,L'}$ in terms of $\sigma_{G,L'}$. By Lemma 3.10, $c_\Phi \geq c_{\Psi'}$, where $\Psi' = \pi_{G,L'}(\Phi)$. (This is useful only if $c_{\Psi'} > 1$.) The main case where $c_{\Psi'} = 1$ (and

hence $c_\Psi = 1$) is when Ψ is of defect 0. Corollary 5.7 below tells us that if $\Phi \neq St$ then Ψ is not of defect 0 for some maximal parabolic subgroup of G . In order to prove this, we first turn Theorem 2.1 to a shape which allows to control $\text{Soc}(S|_P)$ in terms of S , where $S \in \text{Irr } G$. This is necessary mainly for twisted Chevalley groups.

Let \mathbf{G} be an algebraic group over F of rank n , $\alpha_1, \dots, \alpha_n$ be simple roots and $\omega_1, \dots, \omega_n$ be the fundamental weights of \mathbf{G} . Let D denote the Dynkin diagram of \mathbf{G} with nodes labeled by $1, \dots, n$ according to Bourbaki [2]. We denote by X_α the root subgroup of \mathbf{G} corresponding to a root α . The dominant weights of \mathbf{G} are of shape $\sum a_i \omega_i$ for some integers $a_1, \dots, a_n \geq 0$; for an integer q , those with $0 \leq a_i \leq q-1$ for $i = 1, \dots, n$ are called q -restricted. The irreducible representations of \mathbf{G} are parametrized by the dominant weights. Given a dominant weight μ , we denote by V_μ the irreducible representation of \mathbf{G} corresponding to μ ; this weight μ is called the highest weight of V_μ . For our purpose we may assume that \mathbf{G} is simply connected.

Let $G = \mathbf{G}^{Fr}$. Usually one takes for q the common absolute value of Fr acting on the weight lattice of \mathbf{G} , and set $G(q) = \mathbf{G}^{Fr}$. If q is an integer then $G = G(q) \in \{SL(n+1, q), SU(n+1, q), Sp(2n, q), \text{Spin}(2n+1, q), \text{Spin}^\pm(2n, q), E_i(q), i = 6, 7, 8, {}^2E_6(q), F_4(q), G_2(q), {}^3D_4(q)\}$. Otherwise, q^2 is an integer, and then $G \in \{{}^2B_2(q), {}^2F_4(q), {}^2G_2(q)\}$.

The irreducible representations of G are parameterized by the dominant weights satisfying certain conditions. More precisely, every irreducible representation of G is the restriction to G of an irreducible representation of \mathbf{G} whose highest weight belongs to the set $\Delta(G)$, defined as follows:

$$\Delta(G) = \begin{cases} a_1\omega_1 + \dots + a_n\omega_n : 0 \leq a_1, \dots, a_n < q & \text{if } q \text{ is an integer,} \\ a_i < q\sqrt{1/p} \text{ if } \alpha_i \text{ long, and } a_i < q\sqrt{p} \text{ if } \alpha_i \text{ short} & \text{if } q \text{ is not an integer.} \end{cases}$$

We refer to the elements of $\Delta(G)$ as dominant weights for G . (These are called the basic weights for G in [21, 2.8.1].) By [40, Theorem 43] $\Delta(G)$ parameterizes the irreducible representations of G up to equivalence. Therefore, there is a bijection $\Delta(G) \rightarrow \text{Irr } G$, so the irreducible representations of G can be written as ϕ_λ for $\lambda \in \Delta(G)$. Thus, ϕ_λ extends to a unique irreducible representation of \mathbf{G} with highest weight $\lambda \in \Delta(G)$, see [40]. For brevity we refer to λ as the highest weight of ϕ_λ .

Furthermore, there is a unique weight $\lambda \in \Delta(G)$ with maximal sum $a_1 + \dots + a_n$ (if $q \in \mathbb{Z}$ then $a_1 = \dots = a_n = q-1$). For this λ $\dim \phi_\lambda$ is greater than for all other weights in $\Delta(G)$, and equals $|G|_p$, see [40, Corollary of Theorem 46] or [41, p. 88]. The corresponding FG -module is called here *the Steinberg module*, and is denoted by St . We record this as follows:

If q is not an integer, then this is refined as follows. Set $q_1 := q/\sqrt{p}$; then q_1 is an integer.

Lemma 5.2. *Define a weight σ as follows: $\sigma = (q-1)(\omega_1 + \dots + \omega_n)$ if q is integer, otherwise and $\sigma = (q_1-1)\omega_1 + (2q_1-1)\omega_2, (3q_1-1)\omega_1 + (q_1-1)\omega_2, (q_1-1)(\omega_1 + \omega_2) + (2q_1-1)(\omega_3 + \omega_4)$, where $q_1 := q/\sqrt{p}$, respectively, for the group $G = {}^2B_2(q), {}^2G_2(q), {}^2F_4(q)$.*

Then $\dim V_\sigma = |G|_p$ and the restriction of V_σ to G is a unique irreducible FG -module of defect 0.

A standard result of the representation theory of finite groups implies that St is a unique irreducible FG -module of defect 0, and lifts to characteristic 0. It follows that there is a unique irreducible character of G of degree divisible by $|G|_p$. This is called *the Steinberg character*; usually we keep the notation St for this character as well.

For a reductive algebraic group \mathbf{G} Smith's theorem [36] states that if \mathbf{P} is a parabolic subgroup of \mathbf{G} with Levi subgroup \mathbf{L} and V is a rational \mathbf{G} -module then $C_V(O_p(\mathbf{P}))$ is an irreducible \mathbf{L} -module. Furthermore, suppose that the Frobenius endomorphism stabilizes \mathbf{P} and \mathbf{L} , and set $P = \mathbf{P}^{Fr}$, $L = \mathbf{L}^{Fr}$. Then $C_V(O_p(\mathbf{P})) = C_V(O_p(P))$ and this is an irreducible FL -module, see Cabanes [4, 4.2].

To every subset $J \subset D$ one corresponds a parabolic subgroup \mathbf{P}_J by the condition $X_{\alpha_i} \in \mathbf{P}_J$ for $i \in D$ and $X_{-\alpha_i} \in \mathbf{P}_J$ if and only if $i \in J$. (These \mathbf{P}_J are called standard parabolic subgroups. If J is empty, \mathbf{P}_J is a Borel subgroup.) Note that for a subset $J' \in D$ the inclusion $\mathbf{P}_J \subset \mathbf{P}_{J'}$ holds if and only if $J \subset J'$; in particular, every \mathbf{P}_J contains the standard Borel subgroup. Set $\mathbf{G}_J = \langle X_{\pm\alpha_i} : i \in J \rangle$. Then \mathbf{G}_J is the semisimple component of a Levi subgroup \mathbf{L}_J of \mathbf{P}_J . If \mathbf{P}_J and \mathbf{L}_J are Fr -stable, then so is \mathbf{G}_J . We set $P_J = \mathbf{P}_J^{Fr}$, $L_J = \mathbf{L}_J$ and $G_J = \mathbf{G}_J^{Fr}$; these P_J are called standard parabolic subgroups of G .

The following is known but we have no explicit reference:

Lemma 5.3. *$C_V(O_p(P_J))$ is an irreducible FG_J -module.*

Proof. By Lemma 2.1, $C_V(O_p(P_J))$ is an irreducible FL_J -module, so the claim follows from Lemma 3.14. \square

There is some advantage of dealing with \mathbf{G}_J in place of \mathbf{L}_J . The following result is well known [36]:

Lemma 5.4. *Let \mathbf{G} be a simple algebraic group over F , J a non-empty set of nodes at the Dynkin diagram of \mathbf{G} , and $\mathbf{G}_J = \langle X_{\pm\alpha_j} : j \in J \rangle$. Let V be an irreducible \mathbf{G} -module of highest weight ω , and let $v \in V$ be a vector of weight ω . Then $V_J := \langle \mathbf{G}_J v \rangle_F$ is an irreducible direct summand of $V|_{\mathbf{G}_J}$, with highest weight $\omega_J = \sum_{j \in J} a_j \omega_j$.*

If J is connected then \mathbf{G}_J is a simple algebraic group of rank $|J|$, and one may think of the fundamental weights of \mathbf{G}_J as $\{\omega_j : j \in J\}$. Then ω_J means $\sum_{j \in J} a_j \omega_j$. If J is not connected, let $J = J_1 \cup \dots \cup J_k$, where J_1, \dots, J_k are the connected components of J . Then \mathbf{G}_J is the central product of simple algebraic groups $\mathbf{G}_{J_1}, \dots, \mathbf{G}_{J_k}$, where \mathbf{G}_{J_i} corresponds to J_i ($i = 1, \dots, k$). Then it is convenient to us to view ω_J as the string $(\omega_{J_1}, \dots, \omega_{J_k})$. Furthermore, $V_J|_{\mathbf{G}_J}$ is the tensor product of the irreducible representations of \mathbf{G}_{J_i} with highest weight ω_{J_i} for $i = 1, \dots, k$.

There is a version of Lemma 5.4 for finite Chevalley groups. Lemma 2.1 is insufficient as it does not tell us how $(\overline{V}_L)|_{G_J}$ depends on V (in notation of Lemma 5.4). If $G = G(q)$ is non-twisted then this is easy to describe. Indeed, every irreducible FG -module extends to a \mathbf{G} -module with q -restricted highest weight; call it V . Then $G_J := \mathbf{G}_J^{Fr}$ is a non-twisted Chevalley group corresponding to \mathbf{G}_J , and the weight ω_J is q -restricted. Therefore, an irreducible \mathbf{G}_J -module V_J remains irreducible as an FG_J -module, and can be labeled by ω_J . In addition, G_J is the central product of $G_{J_i} := \mathbf{G}_{J_i}^{Fr}$.

This argument can be adjusted to obtain a version for twisted Chevalley group but the twisted group case is less straightforward. The matter is that Fr induces a permutation f , say, of the nodes of the Dynkin diagram of G , which is trivial if and only if G is non-twisted. In the twisted case a set J is required to be f -stable. If every connected component of J is f -stable, then $\omega \in \Delta(G)$ implies $\omega_J \in \Delta(G_J)$. So again we can use ω_J to identify $(V_J)|_{G_J}$.

An additional refinement is required if there is a connected component J_1 , say, of J such that $J_2 := f(J_1) \neq J_1$. If there are roots of different length then, by reordering J_1, J_2 we assume that the roots α_i with $i \in J_1$ are long. Note that the non-trivial f -orbits on $\{1, \dots, n\}$ are of size $a = 2$, except for the case $G = {}^3D_4(q)$ where $a = 3$ [40]. Then $G_J := (\mathbf{G}_{J_1} \circ \mathbf{G}_{J_2})^{Fr} \cong G_{J_1}(q^2) \cong \mathbf{G}_{J_1}^{Fr^2}$ if $a = 2$, or $G_J := (\mathbf{G}_{J_1} \circ \mathbf{G}_{J_2} \circ \mathbf{G}_{J_3})^{Fr} \cong G_{J_1}(q^3) \cong \mathbf{G}_{J_1}^{Fr^3}$ if $a = 3$. Thus, in this case the Chevalley group obtained from the f -orbit on J is non-twisted and quasi-simple. So one would wish to identify the representation $V_J|_{G_J}$ in terms of algebraic group weights of \mathbf{G}_{J_1} rather than of $\mathbf{G}_{J_1} \circ \mathbf{G}_{J_2}$ when $a = 2$, or $\mathbf{G}_{J_1} \circ \mathbf{G}_{J_2} \circ \mathbf{G}_{J_3}$ when $a = 3$. We do this in the following proposition. For this purpose it suffices to assume that f is transitive on the connected components of J . To simplify the language, we call the highest weight of \mathbf{G}_{J_1} obtained in this way the highest weight of V_J .

Proposition 5.5. *Let V be an irreducible \mathbf{G} -module of highest weight $\omega = \sum a_i \omega_i$ such that $V|_G$ is irreducible (so $\omega \in \Delta(G)$). Let J be an f -stable set of nodes at D , the Dynkin diagram of \mathbf{G} . Suppose that J is not connected and f is transitive on the connected components of J . Set $J_i = f^{i-1}(J_1)$ for $1 < i \leq a$ and $\omega_{J_i} = \sum_{j \in J_i} a_j \omega_j$. Then $G_J \cong G_{J_1}(q^a)$.*

Let $\tilde{\omega}_J$ be the highest weight of V_J viewed as a $G_{J_1}(q^a)$ -module. Then $\tilde{\omega}_J \in \Delta(G_{J_1}^{Fr(q^a)})$. More precisely, set $\omega'_{J_1} = \sum_{j \in J_1} a_{f(j)} \omega_j$ and, if $a = 3$ set $\omega''_{J_1} = \sum_{j \in J_1} a_{f^2(j)} \omega_j$. Then:

If q is an integer then $\tilde{\omega}_J = \omega_{J_1} + q\omega'_{J_1}$ for $a = 2$, and $\tilde{\omega}_J = \omega_{J_1} + q\omega'_{f(J_1)} + q^2\omega''_{f^2(J_1)}$ for $a = 3$;

If q is not an integer then $q^2 = p^{2e+1}$ for some integer $e \geq 0$, and $\tilde{\omega}_J = \omega_{J_1} + p^e \omega_{f(J_1)}$.

Proof. We consider only $a = 2$, as the case $a = 3$ differs only on notation. Thus, we show that $G_J := (\mathbf{G}_{J_1} \circ \mathbf{G}_{J_2})^{Fr} \cong G_{J_1}(q^2)$. Note that Fr permutes \mathbf{G}_{J_1} and \mathbf{G}_{J_2} , and acts as follows. Let $x_i \in \mathbf{G}_{J_i}$ ($i = 1, 2$). If $q = p^e$ is an integer then $Fr(x_1, x_2) = (Fr_0^e x_2, Fr_0^e x_1)$, where Fr_0 is the standard Frobenius endomorphism arising from the mapping $y \rightarrow y^p$ ($y \in F$). If q is not an integer, then $Fr(x_1, x_2) = (Fr_0^{e+1} x_2, Fr_0^e x_1)$.

So Fr^2 stabilizes each \mathbf{G}_{J_i} , and its fixed point subgroup on \mathbf{G}_{J_i} is $G_{J_i}(q^2)$. Then (x_1, x_2) is fixed by Fr if and only if $x_1 \in G_{J_1}(q^2)$ and $x_2 = Fr_0^e(x_1)$. So $(\mathbf{G}_{J_1} \circ \mathbf{G}_{J_2})^{Fr} \cong G_{J_1}(q^2)$, as claimed. Furthermore, $(V_J)|_{\mathbf{G}_{J_1} \circ \mathbf{G}_{J_2}}$ is the tensor product of the irreducible \mathbf{G}_{J_1} - and \mathbf{G}_{J_2} -modules of highest weights ω_{J_1} and ω_{J_2} , respectively (as the groups \mathbf{G}_{J_1} and \mathbf{G}_{J_2} commute elementwise). One can consider the \mathbf{G}_{J_1} -module obtained from $W|_{\mathbf{G}_{J_1} \circ \mathbf{G}_{J_2}}$ via the homomorphism $\mathbf{G}_{J_1} \rightarrow \mathbf{G}_{J_1} \circ \mathbf{G}_{J_2}$ defined by $x_1 \rightarrow (x_1, Fr_0^e(x_1))$. Clearly, this is the tensor product of the \mathbf{G}_{J_1} -modules of highest weights ω_{J_1} and $p^e \omega'_{f(J_1)}$.

If $q \in \mathbb{Z}$ then ω_{J_1} and $\omega'_{f(J_1)}$ are q -restricted, so $\tilde{\omega}_J = \omega_{J_1} + q\omega'_{f(J_1)}$ is q^2 -restricted, and hence belongs to $\Delta(G_{J_1}(q^2))$.

Suppose that $q \notin \mathbb{Z}$. As $V|_G$ is irreducible, $\omega \in \Delta(G)$. This implies that $a_i < p^e$ for $i \in J_1$ and $a_{f(i)} < p^{e+1}$. Then $a_i + p^e a_{f(i)} < q^2$, as required. (So the highest weight of the \mathbf{G}_{J_1} -module in question is $\omega_{J_1} + p^e \omega'_{f(J_1)} \in \Delta(G_{J_1}(q^2))$.) \square

Remark. In Proposition 5.5 J is disconnected, which implies that the BN-pair rank of G is at least 2.

Example. Let \mathbf{G} be of type A_{2n-1+k} , $k = 0, 1$, $G = SU_{2n+k}(q)$ and $J = \{1, \dots, n-1, n+k+1, \dots, 2n-1+k\}$. Then $J_1 = \{1, \dots, n-1\}$ and $G_J \cong SL(n, q^2)$. Let $\mu = a_1\omega_1 + \dots + a_{2n-1+k}\omega_{2n-1+k}$. Then $\tilde{\omega}_J = (a_1 + qa_{2n-1+k})\omega'_1 + \dots + (a_{n-1} + qa_{n+1+k})\omega'_{n-1}$, where $\omega'_1, \dots, \omega'_{n-1}$ are the fundamental weights of $\mathbf{G}_{J_1} = A_{n-1}$.

As above, D denotes the Dynkin diagram of \mathbf{G} and $G = \mathbf{G}^{Fr}$. Recall that the standard parabolic subgroups P_J of G are in bijection with f -stable subsets J of the nodes of D (and $P_J = G$ if and only if $J = D$).

Lemma 5.6. *Let $D = J \cup J'$ be the disjoint union of two f -stable subsets. Then $\text{Irr } G \rightarrow \text{Irr}(G_J \times G_{J'})$ is a bijection. In addition, if $T_J, T_{J'}$ are maximal split tori in $G_J, G_{J'}$, respectively, then $T_J T_{J'}$ is a maximal split torus in G .*

Proof. The first statement follows from the reasoning in [21, p. 79]. The second one follows from [21, Theorem 2.4.7(a)], which tells us that $\Pi_i T_{I(i)}$ is a maximal split torus in G , where $I(i)$ is the f -orbit containing i and $T_{I(i)}$ is a maximal split torus in $G_{I(i)}$.

By induction, it follows that a similar statement is true for any disjoint union $D = \cup_i J_i$ of f -stable subsets J_i of D , in particular, when J_i is an f -orbit for every i . (Only this case is explicitly mentioned in [21].) Note that the bijection in Lemma 5.6 yields a bijection $\text{PIM}_G \rightarrow \text{PIM}_{G_J \times G_{J'}}$.

Corollary 5.7. *$G = \mathbf{G}^{Fr}$ be a Chevalley group of BN-pair rank at least 2, and $G \neq {}^2F_4(q)$. Let V be an irreducible \mathbf{G} -module with highest weight $\mu = \sum a_i \omega_i \in \Delta(G)$. Suppose that $\emptyset \neq J \subseteq D$ is f -stable. Let $P := P_J$ be the standard parabolic subgroup of G corresponding to J , and $L := L_J$ a Levi subgroup of P . Let $V_J = C_V(O_p(P))$.*

- (1) $\dim V_J = 1$ if and only if $a_i = 0$ for all $i \in J$.
- (2) $V_J|_L$ is of defect 0 if and only if $a_i = q - 1$ for all $i \in J$. (The latter is equivalent to saying that $M|_{G_J}$ is Steinberg).
- (3) Let $J' = D \setminus J$ and L' be a Levi subgroup of $P_{J'}$. If $V_J|_L$ and $V_{J'}|_{L'}$ are of defect 0 then so is $V|_G$. Equivalently, If $V_J|_{G_J}$ and $V_{J'}|_{G_{J'}}$ are Steinberg modules then $V|_G$ is Steinberg.

Proof. Let \mathbf{P}_J and \mathbf{G}_J be the respective algebraic groups. Then $V_J|_{G_J} = C_V(O_p(\mathbf{P}_J))$, and V_J is an irreducible \mathbf{G}_J -module.

(1) follows from Lemma 5.4.

(2) Suppose that $a_i = q - 1$ for all $i \in J$. Then $V_J|_{G_J}$ is Steinberg by Lemma 5.2. To prove the converse, observe that if $a_i < q - 1$ for some $i \in J$ then $\dim V_J < \dim St$ by Lemma 5.2 applied to \mathbf{G}_J . As $|L|_p = |G_J|_p$, it follows that V_J is of defect 0. This implies (2).

(3) By (2), $a_i = q - 1$ for all $i \in D$, so the claim follows from Lemma 5.2. \square

Lemma 5.8. *Let M be the Steinberg FG -module for a Chevalley group $G = \mathbf{G}^{Fr}$, and let B be a Borel subgroup of G . Then $\text{Soc}(M|_B) = 1_B$.*

Proof. Let U be the Sylow p -subgroup of B . As M is projective and $\dim M = |U|$, it follows that $M|_U$ is the regular FU -module. Therefore, $\dim \text{Soc}(M|_B) = 1$. Let λ be the Brauer character of $\text{Soc}(M|_B)$. As U is in the kernel of λ , it can be viewed as an ordinary character of B . Let St denote the character of M , so this is exactly the Steinberg character. By Proposition 3.15, $(St, \lambda^G) > 0$. If $\lambda \neq 1_B$ then $(\lambda^G, 1_B^G) = 0$ [13, Theorem 70.15A]. In addition, $(St, 1_B^G) = 1$ [13, Theorem 67.10]. Therefore, $\lambda = 1_B$, as required. \square

Lemma 5.9. *Let $q \in \mathbb{Z}$ and let V_μ be an irreducible \mathbf{G} -module of highest weight $\mu = \sum a_i \omega_i \in \Delta(G)$. Let B be a Borel subgroup of $G = \mathbf{G}^{Fr}$. Then the following are equivalent:*

- (1) $\text{Soc}(V_\mu|_B) = 1_B$.
- (2) $a_1, \dots, a_n \in \{0, q-1\}$ and $a_{f(i)} = a_i$ for every i .

Proof. Note that the result is well known for $G = SL(2, q)$.

Suppose first that $G \cong SU(3, q)$, so $\mathbf{G} \cong SL(3, F)$. Let M be the irreducible module for \mathbf{G} of highest weight ω_1 . Then $M|_G$ is isomorphic to the natural module M' , say, for $G \cong SU(3, q)$. Let \mathbf{B} be a Borel subgroup of \mathbf{G} such that $B = \mathbf{B} \cap G$ is a Borel subgroup of G . As explained in the discussion after Lemma 5.2, the socle of \mathbf{B} on M coincides with the socle of B . Let $v \in M$ be a vector of highest weight ω_1 on M . Then $\langle v \rangle$ is \mathbf{B} -stable and hence B -stable. We can view M' as an $F_{q^2}G$ -module endowed by a unitary form, and choose $v \in M'$. Then v is an isotropic vector. It is easy to see that T_0 is isomorphic to the multiplicative group $F_{q^2}^\times$ of F_{q^2} . Let $\nu \in \text{Irr } T_0$ be the representation of B on $\langle v \rangle$. Then it is faithful. As M^* , the dual of M , has highest weight ω_2 , one can check that the representation of T_0 in $C_{M^*}(B)$ is ν^q . Therefore, the representation of T_0 on $C_{V_\mu}(B)$ is $\nu^{a_1} \nu^{a_2 q}$. It follows that $\text{Soc}(V_\mu|_B) = 1_B$ if and only if $\nu^{a_1} \nu^{a_2 q} = 1_{T_0}$. Let t be a generator of T_0 . Then $\nu^{a_1} \nu^{a_2 q}(t) = t^{a_1 + a_2 q}$. It is clear that $t^{a_1 + a_2 q} = 1$ if and only if either $a_1 = a_2 = 0$ or $a_1 = a_2 = q-1$. This completes the proof for $G = SU(3, q)$.

Other groups G are of BN-pair rank 2, and hence satisfy the assumptions of Corollary 5.7.

(1) \rightarrow (2). Let $i \in D$ be such that $a_i \neq 0, q-1$, and let J be the f -orbit of i . Let P_J and G_J be as in Corollary 5.7, and let $B_J = G_J \cap B$, so B_J is a Borel subgroup of G_J .

Suppose first that $|J| = 1$. Then $J = \{i\}$ for some i , and $G_J = SL_2(q)$ (both in the twisted and non-twisted cases). Then $V_J := (\text{Soc}(V_\mu|_{P_J}))|_{G_J}$ is irreducible with highest weight $a_i \omega_i$. Then $\text{Soc}(V_J|_{B_J}) = 1_{B_J}$ if and only if $a_i \in \{0, q-1\}$. As $1_B = (\text{Soc}(V_\mu|_B))$ equals $\text{Soc}(V_J|_{B_J})$ inflated to B , it follows that $a_i \in \{0, q-1\}$.

Let $f(i) = j$, and i, j are not adjacent. If $|J| = 2$ then $G_J = SL_2(q^2)$ and the highest weight of V_J as an FG_J -module is $a_i + qa_j$. So $a_i + qa_j = q^2 - 1$ or 0 . Therefore, $a_i = a_j \in \{0, q-1\}$. Similarly, if $|J| = 3$ then $G_J = SL_2(q^3)$, so $a_i + qa_{f(i)} + q^2 a_{f^2(i)} = q^3 - 1$ or 0 . This again implies $a_i = a_{f(i)} = a_{f^2(i)} \in \{0, q-1\}$.

Let $f(i) = j$, $f(j) = i$ and i, j are adjacent. Then $G_J \cong SU_3(q)$ as $q \in \mathbb{Z}$. As $\text{Soc}(V_J|_{B_J}) \subseteq \text{Soc}(V_\mu|_B) = 1_B$, it follows by the above that $a_i = a_{f(i)} \in \{0, q-1\}$.

(2) \rightarrow (1). Let $J = \{i : a_i = q-1\}$ and $J' = \{i : a_i = 0\}$. Let T_0 be a maximal torus of B . Then $\text{Soc}(V_\mu|_B) = \text{Soc}(\text{Soc}(V_\mu|_P))|_B$ as $\dim \text{Soc}(V_\mu|_B) = 1$ and $\text{Soc}(V_\mu|_P)$ is irreducible. Let B_J be a Borel subgroup of G_J and $T_J = T_0 \cap B_J$. By Corollary 5.7(2),

$(\text{Soc}(V_\mu|_P)|_{G_J})$ is the Steinberg FG_J -module, so, by Lemma 5.8, $((\text{Soc}(V_\mu|_P)|_{G_J})|_{B_J} = 1_{B_J}$, and hence $((\text{Soc}(V_\mu|_P)|_{B_J})|_{T_J} = 1_{T_J}$.

Set $T_{J'} = T_0 \cap G_{J'}$. Then $T_0 = T_J T_{J'}$, see Lemma 5.6.

Let $B_{J'}$ be a Borel subgroup of $G_{J'} \subset P_{J'}$ containing $T_{J'}$. Then $S := (\text{Soc}(V_\mu|_{P_{J'}})|_{G_{J'}}$ is the trivial $FG_{J'}$ -module, so $S|_{B_{J'}} = 1_{B_{J'}}$, and hence $S|_{T_{J'}} = 1_{T_{J'}}$. Therefore, $(\text{Soc}(V_\mu|_B)|_{T_{J'}} = 1_{T_{J'}}$. It follows that $(\text{Soc}(V_\mu|_B)|_{T_0} = 1_{T_0}$, and then $(\text{Soc}(V_\mu|_B) = 1_B$, as required. \square

Proposition 5.10. *Let V_μ be an irreducible \mathbf{G} -module of highest weight $\mu = \sum a_i \omega_i \in \Delta(\mathbf{G})$, and let $J = \{i : a_i = q - 1\}$. Suppose that q is an integer, J is f -stable and $a_i \in \{0, q - 1\}$ for $i = 1, \dots, n$. Let B be a Borel subgroup of G , and let $P = P_J$ be a parabolic subgroup of G . Let Φ be the PIM with socle $V := V_\mu|_G$ and character χ .*

(1) $\text{Soc}(V_\mu|_B) = 1_B$.

(2) $(\chi, 1_B^G) > 0$.

(3) *Let L be a Levi subgroup of P and $S := \pi_{G,P}(V)$ (which is an FL -module). Then S is of defect 0. Furthermore, let ρ be the character of the lift of S . Then ρ is a constituent of $1_{B \cap L}^L$ and $(\chi, \rho^{\#G}) > 0$.*

Proof. (1) is contained in Lemma 5.9. (2) This follows from Corollary 3.16.

(3) By Corollary 5.7, the $S|_{G_J}$ is the Steinberg module. So S is an FL -module of defect 0, and hence lifts to characteristic 0. So $(\chi, \rho^{\#G}) > 0$ by Corollary 3.16.

Furthermore, by the reasoning in Lemma 5.9, $\text{Soc}(\text{Soc}(V_\mu|_P)|_B)$ coincides with $\text{Soc}(V_\mu|_B) = 1_B$. As $S = (\text{Soc}(V_\mu|_P))|_L$, it follows that $S|_{L \cap B} = 1_{L \cap B}$. By Lemma 3.3, where one takes $M = S$, $H = B \cap L$ and $N = O_p(H)$, the truncation $\bar{\rho}$ is the trivial character. By the Harish-Chandra reciprocity, ρ is a constituent of $1_{B \cap L}^L$. \square

The following lemma will be used in Section 7 in order to determine the PIM's for G of dimension $|G|_p$.

Lemma 5.11. *Let $G = \mathbf{G}^{Fr}$ be a Chevalley group of BN -pair rank at least 2, and if $G = {}^2F_4(q)$ assume $q^2 > 2$. Let n be the rank of \mathbf{G} , and let V be an irreducible $F\mathbf{G}$ -module of highest weight $0 \neq \mu = a_1 \omega_1 + \dots + a_n \omega_n \in \Delta(\mathbf{G})$. Suppose that $V|_G \neq St$ and for every parabolic subgroup P of G the restriction of $\text{Soc}(V|_P)$ to a Levi subgroup L is either projective, or the semisimple part L' of L is of type $A_1(p)$, $A_2(2)$ or ${}^2A_2(2)$ and $\text{Soc}(V|_P)$ is trivial on L' . Then one of the following holds:*

(1) $G = SL(3, p)$ and $\omega = (p - 1)\omega_1$ or $(p - 1)\omega_2$;

(2) $G = Sp(4, p) \cong \text{Spin}(5, p)$ and $\omega = (p - 1)\omega_1$ or $(p - 1)\omega_2$;

(3) $G = G_2(p)$ and $\omega = (p - 1)\omega_1$ or $(p - 1)\omega_2$;

(4) $G = SU(4, p)$ and $\omega = (p - 1)(\omega_1 + \omega_3)$;

(5) $G = {}^3D_4(p)$ and $\omega = (p - 1)(\omega_1 + \omega_3 + \omega_4)$;

(6) $G = SU(5, 2)$ and $\omega = \omega_1 + \omega_4$;

In addition, in all these cases $\text{Soc}(V|_B) = 1_B$.

Proof. As above, for an f -stable subset of nodes of the Dynkin diagram of \mathbf{G} we denote by P_J the corresponding parabolic subgroup of G , by G_J the standard semisimple subgroup of P_J and set $V_J = C_V(O_p(P_J))$ viewed as an FG_J -module.

Suppose first that G is of type ${}^2F_4(q)$, $q^2 > 2$. Then there are two f -orbits on D : $J = \{1, 4\}$ and $J' = \{2, 3\}$. Accordingly, G has two parabolic subgroups $P_J, P_{J'}$. By Proposition 5.5, the highest weight of V_J is $(a_1 + 2^e a_4)\omega'_1$, where ω'_1 is the fundamental weight for \mathbf{G}_J of type A_1 and $2^e = q/\sqrt{2}$. The highest weight of $V_{J'}$ is $a_2\omega'_1 + a_3\omega'_2$, where ω'_1, ω'_2 are the fundamental weights for $\mathbf{G}_{J'}$ of type B_2 .

If $V_{(a_1+2^e a_4)\omega'_1}$ is the Steinberg module for $G_{J'}$ then $a_1 + 2a_4 = q^2 - 1$, whence we have $a_1 = q/\sqrt{2} - 1$, $a_4 = q\sqrt{2} - 1$ (as $0 \leq a_1 < q/\sqrt{2}$, $0 \leq a_4 < q\sqrt{2}$). If the restriction of $V_{a_2\omega'_1+a_3\omega'_2}$ to $G_{J'}$ is of defect 0 then $V_{a_2\omega'_1+a_3\omega'_2}|_{G_{J'}}$ is the Steinberg module, and hence $a_2 = q/\sqrt{2} - 1$ and $a_3 = (2q/\sqrt{2}) - 1$ by Lemma 5.2. Thus, $a_1 = a_2 = q/\sqrt{2} - 1$, and $a_3 = a_4 = (2q/\sqrt{2}) - 1$. By Lemma 5.2, $V|_G$ is the Steinberg module.

So assume that $G \neq {}^2F_4(q)$. By Lemma 5.2, at least one of a_1, \dots, a_n differs from $q - 1$.

Suppose first that $G = G(q)$ is non-twisted of rank 2. In notation of Lemma 5.4 and comments following it, $G_J \cong SL(2, q)$ for $J = \{1\}, \{2\}$, so $q = p$, and all these groups are listed in items (1), (2), (3). In addition, V_J is of highest weight $a_1\omega_1$ or $a_2\omega_1$. So the claim about the weights in (1), (2), (3) follows.

Suppose that $G = G(q)$ is non-twisted of rank greater than 2, and let D be the Dynkin diagram of \mathbf{G} . Then one can remove a suitable edge node from D such that $a_i \neq q - 1$ for some i in the remaining set J of nodes. Then the FG -module V_J is not projective (Corollary 5.7), and hence we have a contradiction, unless $G_J \in \{A_1(p), A_2(2)\}$ and $a_i = 0$ for $i \in J$. In fact, $G_J \neq A_1(p)$ as otherwise $|D| = 2$. If $G_J \cong A_2(2) \cong SL(3, 2)$ then $|D| = 3$ and $G = SL(4, 2)$. However, in this case, taking $J = \{1, 2\}$ and $J' = \{2, 3\}$, one obtains $a_i = 0$ for $i \in J \cap J' = D$. Then $\mu = 0$, which is false.

Suppose that G is twisted. We argue case-by-case.

(i) $G = {}^3D_4(q)$. There are two f -orbits J, J' on D , where $J = \{1, 3, 4\}$, $J' = \{2\}$, and hence $G_{J'} \cong SL(2, q)$ and $G_J \cong SL(2, q^3)$. So the assumption is not satisfied unless $q = p$. This case occurs in item (5). The claim on the weights follows from Corollary 5.7.

(ii) $G = {}^2A_n(q)$, where $n = 2m - 1 > 1$ is odd. Take $J = \{1, \dots, m - 1, m + 1, \dots, 2m - 1\}$. Then $G_J \cong SL(m, q^2)$. By assumption, V_J is Steinberg. So $a_1 = \dots = a_{m-1} = a_{m+1} = \dots = a_{2m-1} = q - 1$. Therefore, $a_m < q - 1$. Take $J = \{m\}$. Then $G_J \cong A_1(q)$, whence $q = p$ and $a_m = 0$. The case $n = 3$ is recorded in (4). Let $n > 3$. Take $J = \{m - 1, m, m + 1\}$. Then $G_J \cong {}^2A_3(q)$, which is a contradiction.

(iii) $G = {}^2A_n(q)$, $n = 2m$ even. Take $J = \{1, \dots, m - 1, m + 2, \dots, 2m\}$. Then $G_J \cong A_{m-1}(q^2)$. Then $a_1 = \dots = a_{m-1} = a_{m+2} = \dots = a_{2m} = q - 1$. Therefore, $a_m a_{m+1} < (q - 1)^2$. Then take $J = \{m, m + 1\}$. Then $G_J \cong {}^2A_2(q)$, and hence $q = 2$, $a_m = a_{m+1} = 0$. The case $m = 2$ is recorded in item (6). Let $m > 2$. Then for $J = \{m - 1, m, m + 1, m + 2\}$ we have $G_J \cong {}^2A_4(2)$, which is a contradiction.

(iv) $G = {}^2D_n(q)$, $n > 3$. Take $J = \{n - 1, n\}$. Then $G_J \cong A_1(q^2)$, and the assumption implies $a_{n-1} = a_n = q - 1$, and hence $a_i < q - 1$ for some $i < n - 1$. Next take $J =$

$\{1, \dots, n-2\}$, so $G_J = SL(n-1, q)$. This implies $n = 4$, $q = 2$ and $a_1 = a_2 = 0$. Finally, take $J = \{2, 3, 4\}$. Then G_J is of type ${}^2A_3(2)$, so this option is ruled out.

(v) $G = {}^2E_6(q)$. Then (using Bourbaki's ordering of the nodes) the orbits of f are $(1, 6)$, $(3, 5)$, (2) , (4) . Take $J = \{1, 3, 4, 5, 6\}$. Then $G_J \cong {}^2A_5(q)$, which implies $a_1 = a_3 = a_4 = a_5 = a_6 = q-1$. So $a_2 < q-1$. Next take $J = \{2, 4\}$. Then $G_J \cong A_2(q)$, and V_J is not trivial. This is a contradiction. \square

We close this section by two results which illustrate the use of Propositions 4.7 and 5.5.

Corollary 5.12. *Let $G \in \{B_n(q), n > 2, C_n(q), n > 1, D_n(q), n > 3, {}^2D_{n+1}(q), n > 2\}$. Let $J = \{1, \dots, n-1\}$. Let Φ be a PIM with socle V and $S = \sigma_{G, G_J}(V)$. Then either S is self-dual or $c_\Phi \geq 2c_\Psi$.*

Proof. Let P_J be the standard parabolic subgroup of G corresponding to J and let $L := L_J$ be the standard Levi subgroup of P_J . Then $G_J \cong SL(n, q)$. Let \overline{W}_L be as in Proposition 4.7. We show first that $|\overline{W}_L| = 2$, and a non-trivial element of \overline{W}_L induces the duality automorphism h on L (that is, if $\phi \in \text{Irr } L$ then ϕ^h is the dual of ϕ). Indeed, we can assume that $T_0 \subset L$, and then $N_G(T_0)$ contains an element g acting on T_0 by sending every $t \in T_0$ to t^{-1} . Let h be the inner automorphism of G induced by the g -conjugation. Then S^h is the dual L -module of S . So the result follows from Proposition 4.7. (One can make this clear by using an appropriate basis of the underlying space V of G , and by considering, for the group G_1 of isometries of V , a matrix embedding $GL(n, q) \rightarrow G_1$ sending every $x \in GL(n, q)$ to $\text{diag}(x, {}^{\text{Tr}}x^{-1})$, where ${}^{\text{Tr}}x$ denotes the transpose of x .)

A similar result holds for the groups ${}^2A_{2n+2}(q), {}^2A_{2n+1}(q)$, where P has to be chosen so that L contains $SL(n, q^2)$. In this case h is the duality automorphism following the Galois automorphism.

Proposition 5.13. *Let $k = 0, 1$ and $G = {}^2A_{2n-1+k} \cong SU(2n+k, q)$. Let $J = \{1, \dots, n-1, n+k+1, \dots, 2n-1+k\}$ so $G_J \cong SL(n, q^2)$. Let $\mu = a_1\omega_1 + \dots + a_{2n-1+k}\omega_{2n-1+k} \in \Delta(G)$, let V_μ be an irreducible \mathbf{G} -module of highest weight μ , $V = V_\mu|_G$ and $V_J = \pi_{G, G_J}$.*

(1) *The highest weight of V_J is $\mu' = (a_1 + qa_{2n-1+k})\omega'_1 + \dots + (a_{n-1} + qa_{n+k+1})\omega'_{n-1}$, where $\omega'_1, \dots, \omega'_{n-1}$ are the fundamental weight of $SL(n, F)$.*

(2) *Let Φ be the PIM with socle V , and Ψ be the PIM for G_J with socle V_J . Then either $c_\Phi \geq 2c_\Psi$, or $a_i = a_{n+i+k}$ for $i = 1, \dots, n-1$.*

Proof. (1) follows from Proposition 5.5. (2) Let $1 \neq w \in \overline{W}_L$. The automorphism induced by w on G_J sends $g \in SL(n, q^2)$ to ${}^{\text{Tr}}g^{-\gamma}$, where γ is the Galois automorphism of F_{q^2}/F_q and ${}^{\text{Tr}}g$ is the transpose of g . Then the mapping $v \rightarrow w(g)v$ ($v \in V_J$) yields an irreducible FG_J -module V'_J of highest weight $\mu' := (qa_{n-1} + a_{n+k+1})\omega'_1 + \dots + (qa_1 + a_{2n-1+k})\omega'_{n-1}$. If V_J is not isomorphic to V'_J then $c_\Phi \geq 2c_\Psi$ by Proposition 4.7. If $V_J \cong V'_J$ then $a_1 + qa_{2n+k-1} \equiv (qa_{n-1} + a_{n+k+1})(\text{mod } q^2), \dots, a_{n-1} + qa_{n+k+1} \equiv (qa_1 + a_{2n+k-1})(\text{mod } q^2)$. As $a_i < q$ for $i = 1, \dots, 2n+k$, it follows that $a_i = a_{n+k+i}$ for $i = 1, \dots, n-1$. \square

6. HARISH-CHANRA INDUCTION AND A LOWER BOUND FOR THE PIM DIMENSIONS

The classical result by Brauer and Nesbitt [3, Theorem 8] (see also [20, Ch.IV, Lemma 4.15]) is probably not strong enough to be used for studying PIM dimension bounds for Chevalley groups.

The following lemma is one of the standard results on representations of groups with BN -pair, see [13, pp.683 - 684] or [40, §14, Theorem 48]:

Lemma 6.1. *Let G be a Chevalley group viewed as a group with BN -pair, B a Borel subgroup and W_0 the Weyl group. Then the induced character 1_B^G is a sum of irreducible characters χ_λ of G labeled by $\lambda \in \text{Irr } W_0$, and the multiplicity of χ_λ in 1_B^G is equal to $\lambda(1)$. In other words, $1_B^G = \sum_{\lambda \in \text{Irr } W_0} \lambda(1) \chi_\lambda$.*

Recall that G is called a Chevalley group if $G = \mathbf{G}^F$, where \mathbf{G} is a simple, simply connected algebraic group.

Proposition 6.2. *Let G be a Chevalley group, B a Borel subgroup of G , U the Sylow p -subgroup of B and T_0 a maximal torus of B . Let W_0 be the Weyl group of G as a group with BN -pair and let d be the minimum degree of a non-linear character of W_0 . Let χ be a character vanished at all $1 \neq u \in U$ such that $(\chi, 1_G) = (\chi, St) = 0$. Suppose that the derived subgroup of W_0 has index 2 and, for every non-trivial character $\beta \in \text{Irr } T_0$, either $|W_0\beta| \geq d$ or $(\chi, \beta_B^G) = 0$, where β_B is the inflation of β to B . Then $\chi(1) \geq d \cdot |G|_p$.*

Proof. By Corollary 3.13, $\chi(1) = (\chi, 1_U^G) \cdot |G|_p$. By Lemma 4.1, $(\chi, 1_U^G) = \sum_{\beta \in B} |W_0\beta| (\chi, \beta_B^G)$. Suppose $\chi(1) < d \cdot |G|_p$. Then, by assumption, $(\chi, 1_U^G) = (\chi, 1_B^G)$.

Let χ_λ be the irreducible constituent of 1_B^G corresponding to $\lambda \in \text{Irr } W_0$, see Lemma 6.1. Let m_λ be the multiplicity of χ_λ in χ . By Lemma 6.1,

$$(\chi, 1_B^G) = \sum_{\lambda \in \text{Irr } W} m_\lambda \cdot \lambda(1) < d.$$

It follows that $m_\lambda = 0$ if $\lambda(1) > 1$. So $(\chi, 1_B^G) = \sum_{\lambda \in \text{Irr } W: \lambda(1)=1} m_\lambda$. It is well known [13, Theorem 67.10] that 1_G and St occur in 1_B^G with multiplicity 1. As W_0 has exactly two one-dimensional representations, χ_λ is either St or 1_G . This is a contradiction, as neither St nor 1_G is a constituent of χ . \square

Remarks. (1) The values of d are given by Table 2. (2) The derived subgroup of W_0 has index 2 if and only if $G \in \{A_n(q), D_n(q), E_6(q), E_7(q), E_8(q), {}^2B_2(q), {}^2G_2(q)\}$. (The last two groups are of BN -pair rank 1, and hence $|W_0| = 2$.)

Proposition 6.3. *Let G be a Chevalley group, B a Borel subgroup of G , and T_0 a maximal torus of B . Let W_0 be the Weyl group of G as a group with BN -pair and let d be the minimum degree of a non-linear character of W_0 . Let $\Phi \neq St$ be a PIM for G with character χ . Suppose that the derived subgroup of W_0 has index 2 and $|W_0\beta| \geq d$ for every non-trivial character $\beta \in \text{Irr } T_0$, where β_B is the inflation of β to B . Then $c_\Phi \geq d$, unless $\Phi = \Phi_1$ and $G = SL(2, p)$, $SL(3, 2)$ or ${}^2G_2(3)$.*

Proof. We show that the hypothesis of Proposition 6.2 holds. Indeed, $\chi(u) = 0$ for every $1 \neq u \in U$, as χ is the character of a projective module. As St is a character of a PIM, St

is not a constituent of χ . Suppose that $(\chi, 1_G) > 0$. Then, by orthogonality relations [20, Ch. IV, Lemma 3.3], we have $\Phi = \Phi_1$ and $(\chi, 1_G) = 1$. So $\dim \Phi_1 = |G|_p$. This contradicts a result of Malle and Weigel [34], unless $G = SL(2, p)$, $SL(3, 2)$ or ${}^2G_2(3)$. With these exceptions, the result now follows from Proposition 6.2. \square

Theorem 6.4. *Let $G \in \{SL(n, q), n > 4, Spin^+(2n, q), q \text{ even}, n > 3, E_6(q), E_7(q), E_8(q)\}$. Let $\Phi \neq St$ be a PIM with socle S .*

- (1) *If $\text{Soc } S|_B \neq 1_B$ then $c_\Phi \geq m$, where $m = n, 2n, 27, 28, 120$, respectively.*
- (2) *Suppose that $\text{Soc } S|_B = 1_B$. Then $c_\Phi \geq d$, where $d = n - 1, n - 1, 6, 7, 8$, respectively.*

Proof. Let W be the Weyl group of G .

(1) By Proposition 4.5, $c_\Phi \geq |W\beta|$. The lower bound m for $|W\beta|$ is provided in Propositions 4.12 and 4.9. In particular, $|W\beta| \geq m$ unless $C_W(\beta) = W$. This implies $\beta = 1_{T_0}$ by Lemma 4.11.

(2) follows from Proposition 6.3. Indeed, let d be the minimum dimension of a non-linear irreducible representation of W ; by Table 2, d is as in the statement (2). As $m > d$, it follows from (1) that $|W\beta| > d$ for every $1_B \neq \beta \in \text{Irr } T_0$. For the groups G in the statement the derived subgroup of W is well known to be of index 2. So the hypothesis of Proposition 6.3 holds, and so does the conclusion. \square

Corollary 6.5. *Let $G = \text{Spin}^+(2n, q)$, $n > 4$, and let $\Phi \neq St$ be a PIM with socle $V = V_\mu|_G$, where V_μ is a \mathbf{G} -module with highest weight $\mu = a_1\omega_1 + \cdots + a_n\omega_n$. Then $c_\Phi \geq n - 1$.*

Proof. Let $J = \{1, \dots, n - 1\}$ or $\{1, \dots, n - 2, n\}$. Then $G_J \cong SL(n, q)$ and the highest weight of V_J is $\nu := a_1\omega_1 + \cdots + a_{n-1}\omega_{n-1}$ or $a_1\omega_1 + \cdots + a_{n-2}\omega_{n-2} + a_n\omega_n$, respectively. So V_J is not the Steinberg FG_J -module at least for one of these two cases. Furthermore, by Theorem 6.4, if $V_J \neq St$ then $c_\Psi \geq n - 1$, where Ψ is a PIM for G_J with socle V_J . By Lemma 3.10, $c_\Phi \geq c_\Psi$, and the result follows. \square

Remark. If $G = SL(n, q)$ or $\text{Spin}^+(2n, q)$ then it follows from Theorem 6.4 and Corollary 6.5 that $c_\Phi \rightarrow \infty$ as $n \rightarrow \infty$, provided $\Phi \neq St$. If $G \in \{Sp(2n, q), \text{Spin}(2n + 1, q), \text{Spin}^-(2n + 2, q)\}$ then a similar argument gives that $c_\Phi \geq n - 1$ unless $a_1 = \cdots = a_{n-1} = q - 1$ (respectively, $c_\Phi \geq n - 2$ unless $a_1 = \cdots = a_{n-2} = q - 1$). However, this does not lead to the same conclusion as above. A similar difficulty arises for the groups $SU(2n, q)$ and $SU(2n + 1, q)$.

Proposition 6.6. *Let $G = G(q)$ be a non-twisted Chevalley group, and let V_μ be a \mathbf{G} -module with highest weight $\mu = \sum a_i\omega_i \in \Delta(G)$. Let Φ be a PIM of G with socle $V_\mu|_G$. Let J be a subset of nodes on the Dynkin diagram of \mathbf{G} , not adjacent to each other. Suppose that $a_i \neq 0, q - 1$ for $i \in J$. Then $c_\Phi \geq 2^{|J|}$.*

Proof. As the nodes of J are not adjacent, it follows that \mathbf{G}_J is the direct product of $|J|$ copies of $G_i \cong SL(2, F)$ for $i \in J$. Furthermore, the Smith correspondent σ_{G, G_J} of V_μ is the tensor product of irreducible FG_i -modules with highest weight $a_i\omega_1$. Let $\Psi = \pi_{G, G_J}(\Phi)$ be the parabolic descent of Φ to G_J . Then $c_\Psi \geq 2^{|J|}$ by Lemmas 7.3, 3.2, and $c_\Phi \geq c_\Psi$ by parabolic descent. \square

The proof of Proposition 6.6 illustrates the fact that the parabolic descent from non-twisted groups $G(q)$ for $q = p$ to minimal parabolic subgroups (distinct from the Borel subgroup) does not work. Indeed, in this case $G_J \cong A_1(p)$; if all coefficients a_i are equal to 0 or $p - 1$ then $c_\Psi = 1$. One observes that it is possible to run the parabolic descent to subgroups $G_J \cong SL(3, p)$, obtaining tensor product of irreducible representations with highest weight $(p - 1)\omega_1$ or $(p - 1)\omega_2$, and then use known results for PIM's with such socles.

7. THE REE GROUPS

In this section we prove Theorem 1.1 for Ree groups $G = {}^2G_2(q)$, $q^2 = 3^{2f+1} > 3$. Note that $|G|_p = q^{12} = 3^{6(2f+1)}$.

Note that if $f = 0$ then $G \cong \text{Aut } SL(2, 8)$. For this group the decomposition numbers are known, and $c_\phi = 1$ if and only if $\Phi = \Phi_1$.

Lemma 7.1. *Let χ be a character of degree q^6 vanishing at all unipotent elements of G . Suppose that $(\chi, 1_G) = 0$. Then $\chi = St$.*

Proof. Suppose the contrary. Then $(\chi, 1_U^G) = 1$ (Corollary 3.13), so there is exactly one irreducible constituent τ of χ common with 1_U^G . Recall that $1_U^G = \sum_{\beta \in \text{Irr } T_0} \beta_B^G$, where β_B is the inflation of β to B . By Proposition 4.4, $\tau \in \beta_B^G$ implies that β is W_0 -invariant. (Or straightforwardly, the degree of every β_B^G is $|G : B| = |G|_p + 1$, so we can ignore the β 's with β_B^G irreducible. Observe that β_B^G is reducible if and only if $C_{W_0}(\beta) = W_0$ [40, Theorem 47].) As T_0 is cyclic of order $q^2 - 1$ and $|W_0| = 2$, the non-identity element of W_0 acts on T_0 as $t \rightarrow t^{-1}$ ($t \in T_0$). So $C_W(\beta) \neq 1$ implies $\beta^2 = 1$. In this case $(\beta_B^G, \beta_B^G) = |C_W(\beta)| = 2$ ([40, Ex.(a) after Theorem 47]), so there are two irreducible constituents. If $\beta = 1_{T_0}$ then $\beta_B^G = 1_G + St$. If $\beta^2 = 1_{T_0} \neq \beta$ then the character table in [44] leaves us with exactly two possibilities for $\tau(1)$, which are $d_1 = q^4 - q^2 + 1$ and $d_2 = q^6 + 1 - d_1 = q^6 - q^4 + q^2$.

The constituents of χ that are not in 1_U^G are cuspidal. (Indeed, every proper parabolic subgroup of G is a Borel subgroup, so every non-cuspidal irreducible character has a 1-dimensional constituent under restriction to U . So they are in 1_U^G .) Malle and Weigel [34, p. 327] recorded the cuspidal character degrees that are less than $|G|_p$, and each degree is greater than $q^6 - d_2$. Therefore, $\tau(1) = d_1$. By Lemma 3.13, there is exactly one regular character γ occurring as a constituent of χ , and γ is not a unipotent character (as St is the only unipotent regular character of G). The two characters listed in [34, p. 327] are unipotent (see [5, p. 463] for the degrees of unipotent characters of G). The remaining regular character degrees are

$$d_3 = (q^2 - 1)(q^4 - q^2 + 1) = q^6 - 2q^4 + 2q^2 - 1, \text{ and}$$

$$d_4 = (q^4 - 1)(q^2 - q\sqrt{3} + 1) = q^6 - q^5\sqrt{3} + q^4 - q^2 + q\sqrt{3} - 1.$$

Thus, $\gamma(1) = d_3$ or d_4 . As $(\chi, 1_U^G) = 1$, we have $(\tau, \chi) = 1$. Note that $q^6 - d_1 - d_3 = q^4 - q^2$. Other characters which may occur in the decomposition of χ are of degree $d_5 = q(q^4 - 1)/\sqrt{3}$, $d_6 = q(q^2 - 1)(q^2 - q\sqrt{3} + 1)/2\sqrt{3}$ or $d_7 = q(q^2 - 1)(q^2 + q\sqrt{3} + 1)/2\sqrt{3}$, see [44]. Each of these degrees is greater than $q^4 - q^2$, so $\gamma(1) = d_4$. So

$$q^6 = (q^4 - q^2 + 1) + (q^6 - q^5\sqrt{3} + q^4 - q^2 + q\sqrt{3} - 1) + \frac{aq(q^4 - 1)}{\sqrt{3}} + \frac{bq(q^2 - 1)(q^2 + q\sqrt{3} + 1)}{2\sqrt{3}} + \frac{cq(q^2 - 1)(q^2 - q\sqrt{3} + 1)}{2\sqrt{3}}, \quad (1)$$

where $a, b, c \geq 0$ are integers, and $a + b + c > 0$.

Cancelling the equal terms and dropping the common multiple $q^2 - 1$, we get

$$a(q^2 + 1) + (q^2 + 1)\frac{b+c}{2} - 3(q^2 + 1) = q\sqrt{3} \cdot \left(\frac{c-b}{2} - 2\right). \quad (2)$$

Suppose first that $a > 2$. Then $(q^2 + 1)\frac{b+c}{2} \leq q\sqrt{3} \cdot \left(\frac{c-b}{2} - 2\right)$, and hence $e := c - b > 0$. Then we have $(q^2 + 1)b + e \cdot \left(\frac{q^2+1}{2} - \frac{q\sqrt{3}}{2}\right) \leq -2q\sqrt{3}$. One easily check that $q^2 + 1 - q\sqrt{3} > 0$, which is a contradiction. So $a \leq 2$.

Let X, Y, m be as in the character table of G in [44], so $m = q/\sqrt{3}$, $|X| = 3$ and $|Y| = 9$. Then one observes from [44] that the irreducible characters of the same degree have the same value at X , and also at Y . Therefore, $\chi(g) = \tau(g) + \gamma(g) + a\xi_5(g) + b\xi_6(g) + c\xi_7(g) = 0$, where $g \in \{X, Y\}$ and $\xi_j(g)$ is the value of any character of degree d_j ($j = 5, 6, 7$). By [44], $\tau(Y) = -\gamma(Y) = 1$, $\xi_5(Y) = -m$, $\xi_6(Y) = \xi_7(Y) = m$. So $a\xi_5(g) + b\xi_6(g) + c\xi_7(g) = -am + (b+c)m = 0$, whence $a = b + c$. So (2) simplifies to

$$3(q^2 + 1)((b + c - 2) = q\sqrt{3} \cdot (c - b - 4). \quad (3)$$

Recall that $a = b + c \leq 2$, and hence $b + c \neq 0$. If $b + c = 2$ then $c = b + 4$, which is a contradiction. If $b + c = 1$ then the left hand side in (3) is not divisible by 9, and hence $q\sqrt{3} = 3$. This is not the case. \square

Proposition 7.2. *Let $G = {}^2G_2(3^{2f+1})$, $q^2 = 3^{2f+1}$. Let $\Phi \neq St$ be a PIM, and let χ be the character of Φ . Then $\chi(1) > q^6$.*

Proof. If $(\chi, 1_G) = 0$, the result follows from Lemma 7.1. Otherwise $\Phi = \Phi_1$, and the result for this case follows from [34]. \square

Proposition 7.2 together with certain known results implies Theorem 1.1 for groups of BN-pair rank 1. These are $A_1(q)$, ${}^2A_2(q)$, ${}^2B_2(q)$, ${}^2G_2(q)$. The groups ${}^2G_2(q)$ have been treated above. Below we quote known results on minimal PIM dimensions for the remaining groups of BN-pair rank 1.

Lemma 7.3. [38] *Let $G = SL(2, q)$, $q = p^k$, and let $\Phi \neq St$ be a PIM of G with socle $V_{m\omega_1}|_G$, where $V_{m\omega_1}$ is an irreducible $SL(2, F)$ -module with highest weight $m\omega_1$ for $0 < m < q - 1$. Let $m = m_0 + m_1p + \dots + m_{k-1}p^{k-1}$ be the p -adic expansion of m , and r the number of digits m_i distinct from $p - 1$. Then $c_\Phi = 2^r$. In addition, $c_{\Phi_1} = 2^k - 1$. In particular, if $q = p$ then $c_\Phi = 2$ unless $\Phi = \Phi_1$ with $c_{\Phi_1} = 1$.*

Lemma 7.4. [6] *Let $G = {}^2B_2(q)$, $q^2 > 2$, and let $\Phi \neq St$ be a PIM of G . Then $c_\Phi \geq 4$.*

Lemma 7.5. *Let $G = SU(3, p)$, $p > 2$, and let $\Phi \neq St$ be a PIM of G . Then $c_\Phi \geq 3$.*

Proof. If $p \equiv 1 \pmod{3}$ then the PIM dimensions are listed in [24, Table 1]. The same holds if $p = 3$, see the GAP library. Let $p \equiv 2 \pmod{3}$. Then the formulas in [23, p. 10], lead to the same conclusion.

Proposition 7.6. *Theorem 1.1 is true for groups of BN-pair rank 1.*

Proof. The result follows from Proposition 7.2 and Lemmas 7.3, 7.4 and 7.5. \square

8. GROUPS $SU(4, p)$ AND ${}^3D_4(p)$

In this section we prove Theorem 1.1 for the above groups.

8.1. Some general observations. Prior dealing with the cases where $G = SU(4, p)$ and ${}^3D_4(p)$, we make some general comments which facilitate computations. These are valid for q in place of p , so we do not assume q to be prime until Section 8.2.

Let \mathbf{H} be a connected reductive algebraic group and $H = \mathbf{H}^{Fr}$. The Deligne-Lusztig theory partitions irreducible characters of H to series usually denoted by \mathcal{E}_s , where s runs over representatives of the semisimple conjugacy classes of the dual group H^* . (See [15, p. 136], where our \mathcal{E}_s are denoted by $\mathcal{E}(\mathbf{G}^F, (s)_{\mathbf{G}^{*F^*}})$.) The duality also yields a bijection $T \rightarrow T^*$ between maximal tori in H and H^* such that T^* can be viewed as $\text{Irr } T$, the group of linear characters of T . If $H = U(n, q)$ or ${}^3D_4(q)$ then $H^* \cong H$.

Lemma 8.1. *Let H be a finite reductive group, B a Borel subgroup of H and $T_0 \subset B$ a maximal torus. Let $B = T_0 U$, where U is the unipotent radical of B . Let $\beta \in \text{Irr } T_0$, and β_B its inflation to B . Let $s \in T_0^* \subset H^*$ be the element corresponding to β under the duality mapping $\text{Irr } T_0 \rightarrow T_0^*$.*

(1) \mathcal{E}_s contains the set of all irreducible constituents of β_B^H . In addition, if an irreducible character χ of H is a constituent of 1_U^H then $\chi \in \mathcal{E}_s$ for some $s \in T_0^*$.

(2) β_B^H contains a regular character and a semisimple character of \mathcal{E}_s .

(3) Suppose that \mathcal{E}_1 coincides with the set of all irreducible constituents of 1_B^G and $s \in Z(H^*)$. Then \mathcal{E}_s coincides with the set of all irreducible constituents of β_B^G .

Proof. (1) By [5, Proposition 7.2.4], $\beta_B^H = R_{T_0, \beta}$, where $R_{T_0, \beta}$ is a generalized character of H defined by Deligne and Lusztig (called usually a Deligne-Lusztig character). By definition [15, p. 136], \mathcal{E}_s contains the set $\{\chi \in \text{Irr } H : (\chi, R_{T_0, \beta}) \neq 0\}$, which coincides with $\{\chi \in \text{Irr } H : (\chi, \beta_B^H) \neq 0\}$. This implies the first claim. (Note that this statement is a special case of [31, Proposition 4.1].) The additional claim follows as $1_U^H = \sum_{\beta \in \text{Irr } T} \beta_B^H$.

(2) This is a special case of [31, Proposition 5.1 and Corollary 5.5].

(3) By [15, Proposition 13.30], there exists a one-dimensional character \hat{s} of H such that the characters of \mathcal{E}_s are obtained from those of \mathcal{E}_1 by tensoring with \hat{s} , and $\hat{s}|_{T_0} = \beta$. It follows that \mathcal{E}_s is the set of all irreducible constituents of $1_B^H \otimes \hat{s}$. As $\hat{s}|_B = \beta_B$, we have $1_B^H \otimes \hat{s} = \beta_B^H$, as required. \square

Remark. (3) can be stated by saying that if \mathcal{E}_1 coincides with the Harish-Chandra series of 1_B^H and $s \in Z(H^*)$ then \mathcal{E}_s coincides with the Harish-Chandra series of β_B^H .

Lemma 8.2. (1) Let $s \in G^*$ be a semisimple element and \mathcal{E}_s the corresponding Lusztig class. Let Γ be a Gelfand-Graev character of G . Then \mathcal{E}_s contains exactly one character common with Γ .

(2) Suppose that $s \in T_0^* = \text{Irr } T_0 \cong T_0$. Let $\beta \in \text{Irr } T_0$ correspond to s . Then every regular character of \mathcal{E}_s is a constituent of β_B^G .

Proof. (1) is well known if \mathbf{G} has connected center. In general this follows from results in [15, Section 14]. Indeed, the character $\chi_{(s)}$ introduced in [15, 14.40] is a sum of regular characters [15, 14.46]. By definition of $\chi_{(s)}$, its irreducible constituents belong to \mathcal{E}_s . In addition, every regular character of G is a constituent of $\chi_{(s)}$ [15, 14.46]. Therefore, every regular character of \mathcal{E}_s is a constituent of $\chi_{(s)}$. As $(\chi_{(s)}, \Gamma) = 1$ [15, 14.44], the claim follows.

(2) By [5, 7.4.4], $\beta_B^G = R_{\mathbf{T}_0, \beta}$, where $R_{\mathbf{T}_0, \beta}$ is the Deligne-Lusztig character. Note that $(R_{\mathbf{T}_0, \beta}, \Gamma) = 1$ (indeed, it follows from the reasoning in the proof of Lemma 14.15 in [15, 14.44] that $(R_{\mathbf{T}_0, \beta}, \Gamma) = \pm 1$; as β_B^G is a character, $(\beta_B^G, \Gamma) \geq 0$). (This also follows from [32, Proposition 2.1].) So the claim follows from (1). \square

Lemma 8.3. Let $G = SU(4, q)$, resp., ${}^3D_4(q)$, and let $J = \{1, 3\}$, resp., $\{1, 3, 4\}$ be a set of nodes on the Dynkin diagram of the respective algebraic group \mathbf{G} . Let $\mu = (q-1) \sum_{j \in J} \omega_j$, V_μ an irreducible \mathbf{G} -module of highest weight μ , and $V = V_\mu|_G$. Then $S := \sigma_{G, L}(V)$ is of defect 0. Let Ψ be a PIM with socle S and let λ be the character of the lift of S . Then $(\lambda^{\#G}, \chi) > 0$. In addition, if $c_\Phi = 1$ then $(\lambda^{\#G}, \tau) = 1$.

Proof. The statement about the defect of S and the inequality $(\lambda^{\#G}, \chi) > 0$ is proved in Proposition 5.10(3). Moreover, it is shown there that λ is a constituent of $1_{B \cap L}^L$. Therefore, by transitivity of Harish-Chandra induction [13, Proposition 70.6(iii)], $1_B^G = \lambda^{\#G} + \lambda'$, where λ' is some character of G . So τ is a common constituent of $\lambda^{\#G}$ and 1_U^G ; as $(\tau, 1_U^G) = 1$, it follows that $(\tau, \lambda^{\#G}) = 1$. \square

Lemma 8.4. Let $H = U(4, q)$, B a Borel subgroup of H and $T_0 \subset B$ a maximal torus. Let $B = T_0 U$, where U is the unipotent radical of B . Let $\beta \in \text{Irr } T_0$, and β_B its inflation to B . Let $s \in T_0^* \subset H^*$ be the element corresponding to β under the duality mapping $\text{Irr } T_0 \rightarrow T_0^*$.

(1) \mathcal{E}_1 coincides with the set of all irreducible constituents of 1_B^H .

(2) \mathcal{E}_s coincides with the set of all irreducible constituents of β_B^H .

Proof. (1) Note that H has no cuspidal unipotent irreducible character, see [5, p. 457]. It follows that every unipotent character of H is a constituent of 1_B^G . This is equivalent to the statement.

(2) Observe that $H \cong H^*$. In notation of [35], A_1, A_9, B_1, C_1, C_3 are the only conjugacy classes that meet T_0^* . If $s \in A_1$ then $s \in Z(H^*)$, and the claim follows Lemma 8.1(2). If s belongs to B_1, C_1 , or C_3 then \mathcal{E}_s contain only regular and semisimple characters, so the claim follows from (2). Let $s \in A_9$. By [15, 13.23], $|\mathcal{E}_s|$ is equal to the number of unipotent characters of $C_{H^*}(s)$. This group is isomorphic to $U(2, q) \times U(2, q)$, so the number of unipotent characters equals 4. By Lemma 8.1(1), the irreducible constituents of β_B^H are contained in \mathcal{E}_s . We show that β_B^H has 4 distinct irreducible constituents.

Indeed, $(\beta_B^H, \beta_B^H) = |W_s|$, where W_s is the stabilizer of s in W_0 (see Section 4 for the definition of W_0). It is easy to observe that W_s is an elementary abelian group of order 4. By (2), the regular and semisimple characters occur as constituents of β_B^H . As $(\beta_B^H, \beta_B^H) = 4$ is the sum of squares of the multiplicities of the irreducible constituents of β_B^H , it follows that there are exactly four distinct constituents. This equals $|\mathcal{E}_s|$, the result follows. \square

Let Φ be a PIM for G with character χ . Suppose $c_\Phi = 1$. Then, by Corollary 3.13, $(\chi, 1_U^G) = (\chi, \Gamma) = 1$, which means that χ has exactly one irreducible constituent common with 1_U^G , and exactly one irreducible constituent common with every Gelfand-Graev character Γ . Denote them as τ and γ , respectively. By Proposition 5.10, $(\chi, 1_B^G) > 0$; as $(\chi, 1_U^G) = 1$, it follows that τ is constituent of 1_B^G . (This also follows from Proposition 4.4.) Recall that the irreducible constituents of Γ are called regular characters in the Deligne-Lusztig theory. In general there are several Gelfand-Graev characters, however, if $G = U(n, q)$ or ${}^3D_4(q)$ then G has a single Gelfand-Graev character (see [15, 14.29]). Note that by Lemma 3.14, it suffices to prove that $c_\Phi > 1$ for the group $U(4, q)$ in place of $SU(4, q)$.

In order to prove Theorem 1.1 for the groups $G = SU(4, p)$ and ${}^3D_4(p)$ we first determine $\tau(1)$ and $\gamma(1)$, and observe that $\tau \neq \gamma$ by Lemma 8.1. Next we express $\chi = \tau + \gamma + \sum \nu_i$, where ν_i runs over the characters that are neither regular nor in 1_U^G . In particular, $\nu_i(1) \leq \chi(1) - \tau(1) - \gamma(1)$. As the character table of G is available in [35] for $U(4, q)$ and in [15] for ${}^3D_4(q)$, we obtain a contradiction by inspecting all the possibilities.

The reasoning below does not use much from modular representation theory. In fact, we prove the following. Let G be either $SU(n, q)$ or ${}^3D_4(p)$, and let χ be an ordinary character vanishing at all non-semisimple elements of G . Let L be a Levi subgroup of G whose semisimple part is isomorphic to $SL(2, q^a)$, where $a = 2$ for the former group and $a = 3$ for the latter one. Suppose that $\bar{\chi}_L$ is of defect zero. Then $\chi(1) > |G|_p$. (Of course one has to replace p by q , and use Lemma 3.12 in place of Corollary 3.13.)

8.2. The unitary groups $G = SU(4, p)$. Note that $|G| = p^6(p^4 - 1)(p^3 + 1)(p^2 - 1)$. Let Φ be a PIM with socle V and character χ . Arguing by contradiction, we suppose that $\dim \Phi = |G|_p$, that is, $c_\Phi = 1$.

Let $J = \{1, 3\}$ be a set of nodes at the Dynkin diagram of $\mathbf{G} = SL(4, F)$. Then $G_J \cong SL(2, p^2)$, see Example prior Lemma 5.6. Let $P = P_J$ be a parabolic subgroup of G corresponding to J and $L = L_J$ its Levi subgroup. Let $S = \sigma_{G, P_J}(V)$ be the Smith-Dipper correspondent of V . By Corollary 5.7, $S|_{G_J}$ is the Steinberg module. So S is an FL -module of defect 0, and hence lifts to characteristic 0. Let λ be the character of the lift so $\lambda(1) = p^2$. By Lemma 8.3, τ is a constituent of $\lambda^{\#G}$.

In order to determine $\tau(1)$ we first decompose $\lambda^{\#G}$ as a sum of irreducible constituents.

Lemma 8.5. (1) $\lambda^{\#G} = St + \sigma + \sigma'$, where σ, σ' are irreducible characters of G such that $\sigma(1) = p^2(p^2 + 1)$ and $\sigma'(1) = p^3(p^2 - p + 1)$.

(2) Let τ be a common irreducible constituent for χ and 1_U^G . Then $\tau = \sigma'$, where σ' is as in (1). In particular, $\tau(1) = p^3(p^2 - p + 1)$.

Proof. (1) The degrees of irreducible constituents of 1_B^G are given in [10, Proposition 7.22]. As $\lambda^{\#G}(1) = p^2|G : P_J| = p^2(p+1)(p^3+1)$, this implies (1).

(2) By Corollary 3.13, $(\chi, 1_U^G) = 1$; as mentioned in [10, Proposition 7.22], σ occurs in 1_B^G with multiplicity 2, so $(\chi, \sigma) = 0$, and hence $\tau = \sigma'$, as stated. \square

We need to write down the degrees of the regular characters of $H := U(4, p)$. By the Deligne-Lusztig theory, the regular characters of H are in bijection with semisimple conjugacy classes in the dual group $H^* \cong H = U(4, p)$. So we write ρ_s for the regular characters of H corresponding to s , a representative of a semisimple conjugacy classes in H . Furthermore, $\rho_s(1) = |C_H(s)|_p \cdot |G : C_H(s)|_{p'}$.

The irreducible characters of H can be partitioned in classes consisting of characters of equal degree. This has been done in Nozawa [35], who computed the irreducible characters of H . Similarly, the elements $g \in H$ can be partitioned in classes consisting of all elements g' such that $C_H(g')$ is conjugate to $C_H(g)$. Below we use Nozawa's notation $A_1, \dots, A_{14}, B_1, \dots$ for such classes. (So each class in question is a union of conjugacy classes.) In Table 4 below the first column lists the semisimple conjugacy classes of H with conjugate centralizers $C_H(s)$, and the second column lists $|C_H(s)|$. In order to extract from [35] the regular characters we use the above formulas for their degrees. The third column lists $\rho_s(1)$. For reader's convenience we also identify ρ_s with notation in [35] in the fourth column. For instance, $\chi_{11}(s)$ for $s \in A_1$ are characters of the same degree p^6 depending on a parameter s . In computations below we do not use s to parameterize the characters; instead we write $\chi_{11}(1) = p^6$ to tell that *every* character from the set χ_{11} takes value p^6 at $1 \in H$. (Note that the degrees of regular characters of $U(4, q)$ are as in Table 4 with q in place of p .)

TABLE 4: Degrees of the regular characters of $H = U(4, p)$

s	$ C_H(s) $	$\rho_s(1)$	ρ_s in [35]
A_1	$ H $	p^6	$\chi_{11}(s)$
A_6	$p^3(p+1)^2(p^2-1)(p^3+1)$	$p^3(p-1)(p^2+1)$	$\chi_{13}(s)$
A_9	$p^2(p+1)^2(p^2-1)^2$	$p^2(p^2+1)(p^2-p+1)$	$\chi_{20}(s)$
A_{12}	$p(p+1)^3(p^2-1)$	$p(p-1)(p^2-p+1)(p^2+1)$	$\chi_{15}(s)$
A_{14}	$(p+1)^4$	$(p-1)^2(p^2-p+1)(p^2+1)$	$\chi_{10}(s)$
B_1	$p(p+1)(p^2-1)^2$	$p(p^2+1)(p^3+1)$	$\chi_8(s)$
B_3	$(p+1)^2(p^2-1)$	$(p-1)(p^2+1)(p^3+1)$	$\chi_6(s)$
C_1	$p^2(p^2-1)(p^4-1)$	$p^2(p+1)(p^3+1)$	$\chi_4(s)$
C_3	$(p^2-1)^2$	$(p+1)(p^2+1)(p^3+1)$	$\chi_2(s)$
D_1	$(p+1)(p^3+1)$	$(p^2-1)(p^4-1)$	$\chi_9(s)$
E_1	p^4-1	$(p+1)(p^3+1)(p^2-1)$	$\chi_5(s)$

Lemma 8.6. *Let $G = SU(4, p)$ and $\Phi \neq St$ be a PIM. Then $\dim \Phi > |G|_p$.*

Proof. (1) Let V be the socle and χ the character of Φ . Then $V = V_\mu|_G$, where V_μ is a \mathbf{G} -module with p -restricted highest weight μ . Suppose that $c_\Phi = 1$; then Lemma 5.11 implies that $\mu = (p-1)(\omega_1 + \omega_3)$.

We have explained in Section 8.1 that $\chi = \tau + \gamma + \sum \nu_i$, where τ is constituent of 1_U^G , γ is a regular character and ν_i are some irreducible characters that are neither regular nor in 1_U^G . In particular, $\nu_i(1) \leq \chi(1) - \tau(1) - \gamma(1)$. We keep notation of Section 8.1.

(2) By Lemma 3.14, $\Phi = \Psi|_G$, where Ψ is some PIM for H . In particular, $\dim \Psi = p^6 = |H|_p$. Let χ be the character of Ψ , so $\chi(1) = p^6$. By Corollary 3.13, χ must contain exactly one regular character ρ .

(3) If $(\chi, \gamma) > 0$ for a regular character γ of H then $\gamma(1) = p(p-1)(p^2+1)(p^2-p+1)$ or $(p-1)^2(p^2+1)(p^2-p+1)$.

Indeed, set $f = p^6 - \tau(1) = p^3(p-1)(p^2+1)$. Then $\rho(1) \leq f$. One easily checks that for characters ρ_s in Table 4 $\rho_s(1) > f$ unless $s \in \{A_6, A_{12}, A_{14}\}$.

Observe that $\rho_s(1) = f$ for $s \in A_6$. However, γ cannot coincide with ρ_s for this s , as otherwise $\tau = \tau'|_G$ for some character $\tau' \in \text{Irr } H$ and $\chi = \tau' + \rho_s$; by inspection in [35], there are non-semisimple elements $g \in G \subset H$ such that $\tau(g) + \rho_s(g) \neq 0$ for $s \in A_6$, while $\chi(g) = 0$ as χ is the character of a projective module. Thus, $s \in \{A_{12}, A_{14}\}$, so $\rho_s \in \{\chi_{15}, \chi_{10}\}$, and (3) follows.

(4) Thus, $\gamma \in \{\chi_{10}, \chi_{15}\}$. Note that $\chi_{10}(1) < \chi_{15}(1)$. Set

$$e_1 = f - \chi_{15}(1) = p(p^2+1)(p-1)^2 \text{ and}$$

$$e_2 = f - \chi_{10}(1) = (p^2+1)(p-1)(2p^2-2p+1).$$

Then $\nu_i(1) \leq e_1$ or e_2 . It follows from Lemma 8.4(3) that $\cup_{s \in T_0^*} \mathcal{E}_s = \text{Irr } 1_U^G$. Therefore, $\nu_i \in \mathcal{E}_s$ for some semisimple elements $s \in H^*$ for $s \notin T_0^*$.

We recall some facts of character theory of Chevalley groups.

Observe that A_1, A_9, B_1, C_1, C_3 are the only conjugacy classes that meet T_0 . So $\nu \in \mathcal{E}_s$ and $s \notin \{A_1, A_9, B_1, C_1, C_3\}$. It is well known that $|\mathcal{E}_s| = 1$ if and only if then $(|C_H(s)|, p) = 1$ (this follows for instance from the formulas for regular and semisimple characters in \mathcal{E}_s , see [5, Ch. 8]). So, if $(|C_H(s)|, p) = 1$ then the regular character is the only character of \mathcal{E}_s . This happens if and only if $s \in \{A_{14}, B_3, C_3, D_1, E_1\}$, see Table 4. As ρ is the only regular character that is a constituent of χ , in our case $s \notin \{A_{14}, B_3, C_3, D_1, E_1\}$. We are left with the cases $s \in \{A_6, A_{12}\}$. Let S denote the subgroup of $C_H(s)$ generated by unipotent elements. By the Deligne-Lusztig theory, the characters in \mathcal{E}_s are of degree $d \cdot |H : C_H(s)|_{p'}$, where d is the degree of a unipotent character of S . If $s \in A_{12}$ then $S \cong SL(2, p)$, and if $s \in A_6$ then $S \cong SU(3, p)$, see [35]. The degrees of unipotent characters of these groups are well known to be $1, p$ and $1, p(p-1), p^3$, respectively. Therefore, non-regular characters in \mathcal{E}_s are of degrees $\chi_{19}(1) = (p-1)(p^2+1)$ and $\chi_{17}(1) = p(p-1)^2(p^2+1)$ for $s \in A_6$, and of degree $\chi_{16}(1) = (p-1)(p^2-p+1)(p^2+1)$ for $s \in A_{12}$. Note that $\chi_{16}(1) > \chi_{17}(1) = e_1 > \chi_{19}(1)$.

For further use, we write down some character values extracted from [35]. Let $g \in A_{10}$, $h \in A_{11}$ with the same semisimple parts, and χ_i are as in the last column of Table 4.

$$(*) \quad \chi_i(g) + (p-1)\chi_i(h) = \begin{cases} 0 & \text{for } i = 10 \text{ and } i = 17 \\ \pm p & \text{for } i = 13 \\ \pm 1 & \text{for } i = 16 \text{ and } i = 19 \end{cases}$$

Note that the absolute value of $\chi_i(g) + (p-1)\chi_i(h)$ is independent from the choice of an individual character in the set χ_i . This is the reason to consider $\chi_i(g) + (p-1)\chi_i(h)$ instead of computing χ at g or h in some formulas below.

Case 1. $\gamma(1) = \chi_{15}(1)$.

The representation τ above is denoted by χ_{13} in [35]. Note that $\chi_{13}(1) + \chi_{15}(1) + \chi_{17}(1) = p^6$. However, χ is not the sum of characters from these sets as $\chi_{13} + \chi_{15} + \chi_{17}$ does not vanish at elements of some class in A_{11} .

It follows that the only possibility is $\chi = \chi_{13} + \chi_{15} + p(p-1)\chi_{19}$. (This is not an actual formula, it only tells that χ is the sum of $p(p-1)$ characters from the set χ_{19} and one character from each set χ_{13} and χ_{15} .) Inspection of [35] shows that this is false. (One can compute the values of these characters at a regular unipotent element of H ; it takes zero values for every character from χ_{13} and χ_{15} and value -1 for every character from χ_{19} . So we have a contradiction.)

Case 2. $\gamma(1) = \chi_{10}(1)$.

$$\begin{aligned} \text{Then } e_2 &= (p^2 + 1)(p-1)(2p^2 - 2p + 1) = x \cdot \chi_{16}(1) + y \cdot \chi_{17}(1) + z \cdot \chi_{19}(1) \\ &= x(p-1)(p^2 - p + 1)(p^2 + 1) + yp(p-1)^2(p^2 + 1) + z(p-1)(p^2 + 1), \end{aligned}$$

where $x, y, z \geq 0$ are integers. Dividing by $(p^2 + 1)(p-1)$, we get:

$$2p^2 - 2p + 1 = x(p^2 - p + 1) + yp(p-1) + z, \text{ or } p(p-1)(2 - x - y) = z + x - 1.$$

An obvious possibility is $x = y = 0, z = 2p^2 - 2p + 1$. Suppose $x + y \geq 1$. If $x + z = 1$ then either $z = 0, x = y = 1$ or $z = 1, x = 0, y = 2$.

Suppose $x + z \neq 1$. Then $0 \neq x + z - 1 \equiv 0 \pmod{p(p-1)}$, so $x + z - 1 = kp(p-1)$ for some integer $k > 0$. This implies $2 = x + y + k$, so $x + y \leq 1$, and hence $x + y = 1, k = 1$. So either $x = 1, y = 0, z = p(p-1)$ or $x = 0, y = 1, z = p(p-1) + 1$.

Altogether we have five solutions.

- (1) $e_2 = (2p^2 - 2p + 1)\chi_{19}(1)$, and $\chi = \chi_{13} + \chi_{10} + (2p^2 - 2p + 1)\chi_{19}$;
- (2) $e_2 = \chi_{16}(1) + \chi_{17}(1)$ and $\chi = \chi_{13} + \chi_{10} + \chi_{16} + \chi_{17}$;
- (3) $e_2 = 2\chi_{17}(1) + \chi_{19}(1)$ and $\chi = \chi_{13} + \chi_{10} + 2\chi_{17} + \chi_{19}$;
- (4) $e_2 = \chi_{16}(1) + (p^2 - p)\chi_{19}(1)$ and $\chi = \chi_{13} + \chi_{10} + \chi_{16} + (p^2 - p)\chi_{19}$;
- (5) $e_2 = \chi_{17}(1) + (p^2 - p + 1)\chi_{19}(1)$ and $\chi = \chi_{13} + \chi_{10} + \chi_{17} + (p^2 - p + 1)\chi_{19}$.

As above, here $(2p^2 - 2p + 1)\chi_{19}$ means the sum of $(2p^2 - 2p + 1)$ characters from the set χ_{19} , and similarly for other cases.

Let $g \in A_{10}, h \in A_{11}$ with the same semisimple parts. Next, we compute $\chi(g) - (p-1)\chi(h)$ for χ in the cases (2) and (3) above. As both g, h are not semisimple, this equals 0. However, computing this for the right hand side in these formulas gives us a non-zero value. This will rule out these two cases.

Case 2: Using (*), we have $0 = \chi(g) + (p-1)(\chi(h) = \chi_{10}(g) + \chi_{13}(g) + \chi_{16}(g) + \chi_{17}(g) + (p-1)(\chi_{10}(h) + \chi_{13}(h) + \chi_{16}(h) + \chi_{17}(h))) = \pm p \pm 1 \neq 0$. This is a contradiction.

Case 3: Similarly, by (*), we have $0 = \chi(g) + (p-1)\chi(h) = \chi_{10}(g) + \chi_{13}(g) + 2\chi_{17}(g) + \chi_{19}(g) + (p-1)(\chi_{10}(h) + \chi_{13}(h) + 2\chi_{17}(h) + \chi_{19}(h)) = \pm p \pm 1 \neq 0$.

In order to deal with cases (1), (4), (5), we compute the character values at the regular unipotent element u (from class A_5 of [35]). We have $\chi_{10}(u) = 1$, $\chi_{13}(u) = 0$, $\chi_{16}(u) = -1$, $\chi_{17}(u) = 0$, $\chi_{19}(u) = 1$. The values do not depend on the choice of an individual character in every class χ_i ($i \in \{10, 13, 16, 17, 19\}$).

Case (1). $\chi(u) = \chi_{10}(u) + \chi_{13}(u) + (2p^2 - 2p + 1)\chi_{19}(u) = 1 - (2p^2 - 2p + 1) \neq 0$, a contradiction.

Case (4). $\chi(u_5) = \chi_{10}(u_5) + \chi_{13}(u_5) + \chi_{16}(u_5) + (p^2 - p)\chi_{19}(u_5) = 1 - 1 - (p^2 - p) \neq 0$, a contradiction.

Case (5). $\chi(u_5) = \chi_{10}(u_5) + \chi_{13}(u_5) + \chi_{17}(u_5) + (2p^2 - 2p + 1)\chi_{19}(u_5) = 1 - (2p^2 - 2p + 1) \neq 0$, a contradiction. \square

Remark. Some characters χ in cases (2) and (3) vanish at non-identity p -elements. (In (3) $2\chi_{17}$ may be the sum of two distinct characters of degree $p(p-1)^2(p^2+1)$.)

8.3. The groups $G = {}^3D_4(p)$. We follow the strategy described in Section 8.1. We can assume $p > 2$ as the decomposition matrix for ${}^3D_4(2)$ is available in the GAP library, and one can read off from there that the minimum value for c_Φ equals 15.

Lemma 8.7. *Let $G = {}^3D_4(p)$, and $\Phi \neq St$ be a PIM. Then $c_\Phi \geq 2$.*

Proof. Let V be the socle and χ the character of Φ . Then $V = V_\mu$, where $\mu = (p-1)(\omega_1 + \omega_3 + \omega_4)$. We keep notation of Section 8.1.

Arguing by contradiction, we assume that $c_\Phi = 1$, and then we denote by τ and γ irreducible characters of G occurring as common constituents of χ with 1_U^G and Γ , respectively.

We first show that τ is a unipotent character of degree $p^7(p^4 - p^2 + 1)$. Let P be a parabolic subgroup of G corresponding to the nodes $J = \{1, 3, 4\}$ at the Dynkin diagram of G . Let L be a Levi subgroup of P and L' the subgroup of L generated by unipotent elements. Then $L' \cong SL(2, p^3)$. Let $S = \text{Soc}(V_\mu|_P)|_L$ and ρ the character of S . By Lemma 8.3, $(\chi, \rho^{\#G}) > 1$. As ρ^G is a part of 1_U^G , it follows that $(\chi, \rho^G) = 1$. In addition, τ is a constituent of 1_B^G . The degrees of irreducible constituents of 1_B^G are given in [10, Proposition 7.22]. The order of G is $p^{12}(p^6 - 1)(p^2 - 1)(p^8 + p^4 + 1)$, so $\rho^{\#G}(1) = p^3|G : P_J| = p^3(p+1)(p^8 + p^4 + 1)$. From this one easily obtains the following lemma (which true for q in place of p):

Lemma 8.8. *In the above notation, $\rho^{\#G} = St + \rho'_1 + \rho'_2 + \rho_2$, where $\rho'_2(1) = p^3(p^3 + 1)^2/2$, $\rho_2(1) = p^3(p+1)^2(p^4 - p^2 + 1)/2$ and $\rho'_1(1) = p^7(p^4 - p^2 + 1)$.*

Lemma 8.9. *Let Φ be a PIM with $c_\Phi = 1$. Then $\tau(1) = p^7(p^4 - p^2 + 1)$.*

Proof. By [10, Proposition 7.22] or [10, p.115], the characters ρ_2, ρ'_2 occur in 1_B^G with multiplicity greater than 1; therefore, none of them is a constituent of χ , and the claim follows. \square

As G coincides with its dual group G^* , we identify maximal tori in G and G^* . Following [14] we denote by s_i , $i = 1, \dots, 15$, the union of the semisimple classes of G with the same

centralizer (that is, $C_G(x)$ is conjugate with $C_G(y)$ for $x, y \in s_i$). In the character table of G in [15] a class with representative $s \in s_i$ meets T_0 if and only if $i \leq 8$. (The set s_8 consists of regular elements.)

Lemma 8.10. *The set $\cup_{i \leq 8} \mathcal{E}_{s_i} \setminus \text{Irr } 1_U^G$ consists of two unipotent cuspidal characters.*

Proof. The unipotent characters of G have been determined by Spaltenstein [37]. All but two of them are constituents of 1_B^G .

The characters $\chi_{3,*}$, $\chi_{5,*}$, $\chi_{6,*}$, $\chi_{7,*}$ and χ_8 from $\cup_{i \leq 8} \mathcal{E}_{s_i}$ are either regular or semisimple. So they are in 1_U^G by Lemma 8.1(2). This leaves with s_i for $i = 2, 4$ (as $s_1 = 1$). In these cases $C_G(s_2) \cong (SL(2, p^3) \circ SL(2, p)) \cdot T_0$ and $C_G(s_4) \cong SL(3, p) \cdot T_0$, where T_0 is a split maximal torus in G , see [14, Proposition 2.2]. For these groups the number of unipotent characters are known to be 4 and 3, respectively. So $|\mathcal{E}_{s_i}| = 4$, resp., 3, for $i = 2, 4$, see [15, Theorem 13.23]. Let $\beta_i \in \text{Irr } T_0$ be the character corresponding to s_i . Then the Harish-Chandra series $\beta_{i,B}^G$ is contained in \mathcal{E}_{s_i} by Lemma 8.1(1). Set $W_i = C_{W_0}(\beta_i)$. Then $W_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $W_4 \cong S_3$ [14, Lemma 3.4]. Moreover, the centralizing algebra of $\beta_{i,B}^G$ is isomorphic to the group algebra of W_i ([40, Exercise in §14 prior Lemma 86]). Therefore, $\beta_{i,B}^G$ consists of 4 distinct irreducible constituents, whereas $\beta_{2,B}^G$ has three distinct irreducible constituents (two of them occurs with multiplicity 1 and one constituent occurs with multiplicity 2). In both the cases $|\mathcal{E}_{s_i}|$ coincides with the number of distinct irreducible constituents in $\beta_{i,B}^G$, and the claim follows. \square

Lemma 8.11. $\gamma(1) \in \{d_1, d_2, d_4\}$, where $d_2 = (p-1)^2(p^3+1)^2(p^4-p^2+1)$, $d_4 = (p-1)(p^3-1)(p^8+p^4+1)$.

Proof. We first write down the degrees of the regular characters of H in Table 5. Note that one can easily detect the regular characters in the character table of G in [14]. Indeed, the characters in [14] are partitioned in Lusztig series \mathcal{E}_s with $s \in s_i$, $i = 1, \dots, 15$. As mentioned in the proof of Lemma 8.2, every \mathcal{E}_s has a single regular character. Then its degree equals $|G : C_G(s)|_{p'} \cdot |C_G(s)|_p$, while other characters in \mathcal{E}_s are of degree $|G : C_G(s)|_{p'} \cdot e$, where e is the degree of a unipotent character in $C_G(s)$. (So the p -power part of the character degrees in \mathcal{E}_s is maximal for the regular character.)

As is explained in Section 8.1, $\gamma(1) \leq p^{12} - \tau(1) = p^{12} - p^{11} + p^9 - p^7$. One observes that for $s \in \cup_{i > 8} s_i$ only the characters $\chi_{12}(s)$ and $\chi_{15}(s)$ satisfy this inequality. So exactly one of these characters is a constituent of χ , as claimed.

Let d_1, d_2 be the degrees of these $\chi_{12}(s)$, $\chi_{15}(s)$, respectively, and set $f_i = p^{12} - p^{11} + p^9 - p^7 - d_i$ ($i = 1, 2$). Then

$$f_1 = p^{11} + p^9 + 4p^8 + p^7 + 2p^6 + 2p^5 + 4p^4 + 2p - 1,$$

$$f_2 = 2p^9 - 2p^8 + p^5 - 2p^4 + p^3 + p - 1 = (p-1)(2p^8 + p^4 - p^3 + 1).$$

Thus, f_i ($i = 1, 2$) is a sum of the degrees of irreducible non-regular characters that do not belong to 1_U^G . Let λ be one of these characters. Note that G has two cuspidal unipotent characters of degrees $e_1 = p^3(p^3-1)^2/2$ and $e_2 = p^3(p-1)^2(p^4-p^2+1)/4$. They do not belong to 1_U^G whereas the other 6 unipotent characters belong to 1_U^G .

Inspection of the character table of G in [14] shows that non-regular characters that are in \mathcal{E}_{s_i} with $i > 8$ are the characters $\chi_{9,1}$, $\chi_{9,qs'}$ and $\chi_{10,1}$ in [14], of degrees $e_3 = (p^3 - 1)(p^2 + p + 1)(p^4 - p^2 + 1)$, $e_4 = p(p^3 - 1)^2(p^4 - p^2 + 1)$ and $e_5 = (p^3 - 1)(p^8 + p^4 + 1)$, respectively. One observes that $(p^2 - p + 1)(p^2 + p + 1) = p^4 - p^2 + 1$ and $(p^2 + p + 1)(p - 1) = p^3 - 1$. As $p^2 + p + 1$ is odd, it follows that $p^2 + p + 1$ divides e_i for every $i = 1, 2, 3, 4, 5$, and hence $p^2 + p + 1$ divides f_j for $j = 1, 2$. However, $f_1 \pmod{p^2 + p + 1} = -5$ and $f_2 \pmod{p^2 + p + 1} = 2p - 11$. This is a contradiction, which completes the proof. \square

TABLE 5: Degrees of the regular characters of $G = {}^3D_4(p)$, $p > 2$

s	$\rho_s(1)$	ρ_s in [15]
s_1	p^{12}	St
s_2	$p^4(p^8 + p^4 + 1)$	$\chi_{2,St,St'}(s)$
s_3	$p^3(p + 1)(p^8 + p^4 + 1)$	$\chi_{3,St}(s)$
s_4	$p^3(p^3 + 1)(p^2 - p + 1)(p^4 - p^2 + 1)$	$\chi_{4,St}(s)$
s_5	$p(p^3 + 1)(p^8 + p^4 + 1)$	$\chi_{5,St}(s)$
s_6	$(p + 1)(p^3 + 1)(p^8 + p^4 + 1)$	$\chi_6(s)$
s_7	$p^3(p - 1)(p^8 + p^4 + 1)$	$\chi_{7,St}(s)$
s_8	$(p - 1)(p^3 + 1)(p^8 + p^4 + 1)$	$\chi_8(s)$
s_9	$p^3(p^3 - 1)(p^2 + p + 1)(p^4 - p^2 + 1)$	$\chi_{9,St}(s)$
s_{10}	$p(p^3 - 1)(p^8 + p^4 + 1)$	$\chi_{10,St}(s)$
s_{11}	$(p + 1)(p^3 - 1)(p^8 + p^4 + 1)$	$\chi_{11}(s)$
s_{12}	$(p - 1)^2(p^3 + 1)^2(p^4 - p^2 + 1)$	$\chi_{12}(s)$
s_{13}	$(p + 1)^2(p^3 - 1)^2(p^4 - p^2 + 1)$	$\chi_{13}(s)$
s_{14}	$(p^6 - 1)^2$	$\chi_{14}(s)$
s_{15}	$(p - 1)(p^3 - 1)(p^8 + p^4 + 1)$	$\chi_{15}(s)$

9. THE MINIMAL PIM'S

In this section we complete the proof of Theorem 1.1. To settle the base of induction, we refer to certain known results for groups of small rank. We first write down some data available in the GAP library:

Lemma 9.1. *Let G be one of the groups below and let Φ be a PIM for G other than the Steinberg PIM.*

- (1) *If $G = Sp(4, 2)$ or $Sp(4, 3)$ then $c_\Phi \geq 3$.*
- (2) *If $G = Sp(4, 3)$ then $c_{\Phi_1} = 2$, and $c_\Phi \geq 3$ for $\Phi \neq \Phi_1$.*
- (3) *If $G = SU(4, 2)$ then $\min c_\Phi \geq 4$.*
- (4) *If $G = SU(5, 2)$ then $\min c_\Phi \geq 5$.*
- (5) *If $G = {}^3D_4(2)$ then $c_\Phi \geq 15$.*
- (6) *If $G = G_2(2)$ then $\min c_\Phi = 5$.*
- (7) *If $G = {}^2F_4(2)$ then $c_\Phi \geq 14$.*

Lemma 9.2. *Let G be one of the groups below and let $\Phi \neq St$ be a PIM for G .*

(1) *Let $G = SL(3, p)$, $p > 2$. Then $c_\Phi \geq 2$.*

(2) *Let $G = Sp(4, p)$, $p > 3$. Then $c_\Phi \geq 3$.*

(3) *Let $G = G_2(p)$, $p > 2$. Then $c_\Phi \geq 6$.*

Proof. The statements (1), (2), (3) follows from the results in [23], [26], [27, Ch. 18], respectively. \square

Proof of Theorem 1.1. Let r be the BN-pair rank of G . If $r = 1$ then the result is contained in Proposition 7.6. In general, suppose the contrary, and let G be a counter example, so $r > 1$. Let $\Phi \neq St$ be a PIM with $c_\Phi = 1$, and let V be the socle of Φ . Then V is irreducible and hence $V = V_\mu|_G$ for some irreducible module V_μ for the respective algebraic group \mathbf{G} . (Here μ is the highest weight of V_μ .) Let P be a proper parabolic subgroup of G , which is not a Borel subgroup of G . Then $P = P_J$ for some non-empty set J , see Section 5. Let L_J be a Levi subgroup of P , let G_J be as in Section 5 and $V_J = C_V(O_p(P))$. Then G_J is a Chevalley group of rank $r_J < r$, and V_J is irreducible both as an FL_J - and an FG_J -module (Lemma 5.3). Let Ψ be the PIM for L_J with socle $V_J|_{L_J}$. By Lemma 3.10, $c_\Psi = 1$, so $\dim \Psi = |L_J|_p$. As $|L_J|_p = |G_J|_p$, the restriction $\Psi|_{G_J}$ is a PIM for G_J of dimension $|G_J|_p$ (Lemma 3.7(2)). This is a contradiction unless $V_J|_{G_J}$ is isomorphic to the Steinberg module for G_J , or else $G_J \in \{A_1(p), A_2(2), {}^2A_2(2)\}$ and $V_J|_{G_J}$ is the trivial FG_J -module. As this is true for every parabolic subgroup of G , which is not a Borel subgroup, V satisfies the assumption of Lemma 5.11. Therefore, G belongs to the list (1) - (6) of Lemma 5.11, or else $G = {}^2F_4(\sqrt{2})$. However, for these groups Theorem 1.1 is true by Lemmas 8.6, 8.7, 9.1 and 9.2, which is a contradiction.

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